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Exact cumulant Kramers–Moyal-like expansion

W.A.M. Morgado

Departamento de Física, Pontifícia Universidade Católica and National Institute of Science and Technology for Complex Systems, 22452-970, Rio de Janeiro, Brazil

HIGHLIGHTS

- Being an exact approach to derive a time-evolution equation for the PDF of a generic system.
- Having a distinct structure of jump-moments from the usual Fokker–Planck or Kramers–Moyal equations.
- Furnishing the time-evolution equation for systems that are not necessarily driven by Langevin-like equations.
- Allowing us to obtain the exact evolution for all the averages and cumulants of the system.

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ABSTRACT

We derive an exact equation, a Cumulant Kramers–Moyal Equation (CKME), quite similar to the Kramers–Moyal Equation (KME), for the probability distribution of a Markovian dynamical system. It can be applied to any well behaved (converging cumulants) continuous time systems, such as Langevin equations or other models. An interesting but significant difference with respect to the KME is that their jump-moments are proportional to cumulants of the dynamical variables, but not proportional to central moments, as is the case for the KME. In fact, they still obey a weaker version of Pawula's theorem, namely Marcinkiewicz's theorem. We compare the results derived from the equations herein with the ones obtained by computing via Gaussian and biased, and unbiased, Poisson Langevin dynamics and a Poisson non-Langevin model. We obtain the exact CKME time-evolution equation for the systems, and in several cases, those are distinct from the Fokker–Planck equation or the KME.

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1. Introduction

The emergence of macroscopic behavior for a system, out of its microscopic degrees of freedom, is one of the most fascinating aspects of physics [1]. The emergent collective behavior presents many universal properties that are not easily guessed from the microscopic components [2–4].

Indeed, the number of microscopic variables is in general so large that it becomes a useless exercise to try and solve the problem of a macroscopic system from first principles. A more useful approach is to reduce the number of variables describing the system. For instance, this can be done when there is a clear time-scale separation that allows us to take averages over the fast-scale variables and then write equations of motion for the slow-scale variables [5,6]. The main goal is that by reducing the number of variables we can still describe the principal aspects of the system and solve it by analytical or numerical means. Let us call this the macroscopic approach.

The basic approaches in describing a system with macroscopic equations are: via averages (as in thermodynamics or hydrodynamics [7]); via probability distributions (as in the Fokker–Planck and other master equations [8]); and via effective equations of motion for selected variables (such as the Langevin equation [9]).

E-mail address: welles@puc-rio.br.

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Let us start from a fundamental microscopic equation, such as Liouville's or Schroedinger's Equation, aiming to reduce its variables to a manageable reduced set. The chosen set must have properties that make it suitable for describing the macroscopic properties of the system. For instance, the hydrodynamic equations are derived for the conserved quantities of the system (mass, momentum and energy) [10]. When a subset of the microscopic variables varies considerably slower than the rest, such as the variables describing a Brownian particle immersed in a thermal bath of lighter particles [11], we can obtain time-evolution equations for the variables describing the Brownian particle by taking averages over the variables of the thermal bath: for the Brownian variables (Langevin equation approach) or a Fokker–Planck Equation (FPE) describing the time-evolution of the probability distribution. Usually, the methods above involve truncating a series on powers of a small dimensionless parameter [12], such as the mass ratio for the bath particle mass over the Brownian particle mass. Whereas the FPE is a second order differential equation there is another equation, namely the Kramers–Moyal Equation [13,14] (KME) that is an infinite order differential equation. In fact, the FPE is just the KME truncated up to second order. Here, we are mainly interested in obtaining a KME-like equation that is completely equivalent to the full dynamics of the system but presents distinct properties for its coefficients than those for the usual KME.

Specifically, we develop a method for obtaining the time-evolution equation for the probability distribution for a general dynamical system, we shall call that equation the Cumulant Kramers–Moyal Equation (CKME). As long as the convergence conditions are obeyed (convergence of cumulants) the detailed character of the system does not affect the final form for that equation.

Some specific examples shall be worked out in the context of Brownian motion. Our approach is generic in principle, but we chose to illustrate the procedures by the use of linear Markovian mechanical models for Gaussian, and non-Gaussian noise. We use a formalism based on the Laplace transform that has been used to obtain exact results for Langevin equations in the context of Markovian and non-Markovian noise [15,16], thermal conductance for linear [17] and non-linear systems [18], fluctuation relations for the work [19], and non Gaussian noises [20,21,18,22]. This method is equivalent to taking into account all noise cumulants, which can be of finite or infinite in number. We also exploit a simple non-Langevin Poisson model and obtain the exact CKME which is neither FPE nor KME.

This paper is organized as follows. In Section 2, we discuss the FPE, and some of its properties. In Section 3 we discuss the Kramers–Moyal Equation (KME) and some of its constraints, such as Pawula's theorem. In Section 4, we obtain a Laplace transformed form for the probability distribution function for a generic well behaved Markovian system. In Section 5, we derive a general Cumulant Kramers–Moyal Equation (CKME) for that Markovian system. In Section 6, we discuss some similarities and dissimilarities between the KME and the CKME. In Section 7, we derive the CKME for Gaussian Langevin systems. In Section 8, we derive it for biased and unbiased Langevin Poisson models. In Section 9, we discuss a simple non-Langevin Poisson model. In Section 10, we discuss the results.

2. The Fokker–Planck equation

The Fokker–Planck Equation (FPE) is a staple of statistical mechanics. Technically, it is a continuous master equation, a second order differential equation that can be applied to many situations of real interest, ranging from finance to plasma physics. It reads in general (x represents the set of variables, such as positions y and velocities v)

$$\partial_t p(x,t) = -\frac{\partial}{\partial x}\left[a^{FP}_{(1)}(x)\ \ p(x,t)\right] + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\left[a^{FP}_{(2)}(x)\ \ p(x,t)\right], \tag{1}$$

where the jump moments (JM) $a^{FP}_{(n)}(x)$ are defined as the central moments [23] below

$$a^{FP}_{(n)}(x) = \lim_{\tau\to 0}\int_{-\infty}^{\infty} \mathrm{d}x'\,\frac{(x'-x)^n}{\tau}\,p(x',t+\tau\mid x,t). \tag{2}$$

The JM can be experimentally determined by either measuring, or calculating, the averages below [23]

$$a_1 = \frac{\langle \Delta x\rangle}{\Delta t}, \quad \text{and} \quad a_2 = \frac{\left\langle \Delta x^2\right\rangle}{\Delta t},$$

for small Δt (where $\Delta x = x' - x$).

An important example of a FPE is the Kramers equation [23]. It governs the probability density for a Brownian particle ($x \equiv (y, v)$) under the action of Gaussian noise and reads

$$\partial_t p(y,v,t) = -v\frac{\partial}{\partial y}p(y,v,t) + \frac{\partial}{\partial v}\left[\left(\frac{\gamma\, v - V'(y)}{m}\right)p(y,v,t)\right] + \frac{\gamma\, T}{m^2}\,\frac{\partial^2}{\partial v^2}p(y,v,t). \tag{3}$$

This is the classical result for the Kramers equation [24] for a particle under a potential $V(y)$. It is straightforward to verify that the equilibrium solution corresponds to the Boltzmann–Gibbs distribution ($k_B = 1$)

$$p_{eq}(y,v) = \frac{\mathrm{e}^{-\frac{mv^2}{2T}-\frac{V(y)}{T}}}{Z}, \quad \text{where } Z = \int \mathrm{d}y\,\mathrm{d}v\,\mathrm{e}^{-\frac{mv^2}{2T}-\frac{V(y)}{T}}. \tag{4}$$

The FPE does not give the correct evolution of the higher order momenta $\langle x^{n\geq 3}\rangle$ unless the distribution is Gaussian. Higher order cumulants are not treated correctly by the Fokker–Planck equation.

However, the FPE preserves normalization and guarantees that probabilities are well defined. A third order differential equation would not even preserve the positivity of the probabilities, except for short times. In fact, we shall see in the next section that a third order FPE (KME) is not even allowed. On the other hand, an infinite order differential equation is allowed.

3. The Kramers–Moyal equation

The Kramers–Moyal Equation (KME) is an extension of the FPE formalism. It is a continuous Master equation, an expansion of the Kolmogorov equation at all orders [23], for a given Markovian system. The KME is defined by

$$\partial_t p(y,v,t) = \sum_{n,m=0,\ n+m\geq 1}^{\infty} \frac{(-1)^n}{n!}\frac{(-1)^m}{m!}\frac{\partial^{n+m}}{\partial y^n\,\partial v^m}\left[a^{KM}_{(n,m)}(y,v)\ \ p(y,v,t)\right], \tag{5}$$

where the jump moments (JM) $a^{KM}_{(n,m)}(y,v)$ are defined as the central moments [23] below

$$a^{KM}_{(n,m)}(y,v) = \lim_{\tau\to 0}\int_{-\infty}^{\infty} dy'\,dv'\ \frac{(y'-y)^n(v'-v)^m}{\tau}\ p(y',v',t+\tau\mid y,v,t). \tag{6}$$

The JM are not completely independent since they obey Pawula's theorem [25,26] which states that whenever a higher order JM ($n+m>2$) vanishes, then all JM above order 2 are zero. Hence, the KME reduces itself to the Fokker–Planck equation. This means that only two forms of KME for the probability distribution are allowed: the FPE (all $a_{n\geq 3}=0$) or the full KME (all $a_{n\geq 3}\neq 0$).

It should be emphasized here that the JM of the KME, Eq. (5), obey a stronger constraint than cumulants in general: they obey PT and either all JM are non-zero, or only the first two of them may be non-zero. As we shall see in the following, cumulants obey a weaker constraint, namely Marcinkiewicz's theorem [27], which allows an infinity of cumulants to exist even if many of them are zero. In fact, we cannot construct a KME with an infinite, but incomplete set of JM, such as the even ones being non-zero, $a^{KM}_{2n}(x)\neq 0$, while the odd ones are zero $a^{KM}_{2n+1}(x)=0$.

Even when $a^{FP}_1 = a^{KM}_1 = a_1$ and $a^{FP}_2 = a^{KM}_2 = a_2$, the prescribed evolution for the momenta can be distinct. So, the same happens for the complete probability distribution function. Let us see the evolution equations for the three first momenta (we assume $a^{KM}_3\neq 0$):

$$\dot{\langle x\rangle}_{KM} = a_1 = \dot{\langle x\rangle}_{FP}, \tag{7}$$

$$\dot{\left\langle x^2\right\rangle}_{KM} = a_2 + 2\,\langle x\,a_1\rangle = \dot{\left\langle x^2\right\rangle}_{FP}, \tag{8}$$

$$\dot{\left\langle x^3\right\rangle}_{KM} = a^{KM}_3 + 3\,\langle x\,a_2\rangle + 6\left\langle x^2\,a_1\right\rangle = a^{KM}_3 + \dot{\left\langle x^3\right\rangle}_{FP}. \tag{9}$$

Thus, whenever the higher order JM exist, which can happen for the case of non-Gaussian noise (Poisson noise presents a fully infinite set of cumulants), using the FPE as an approximation for the dynamics introduces errors in the time evolution of the higher order momenta of the variables. Only the KME shall yield the correct time evolution. However, the FPE still furnishes us with the correct evolution for the first two averages (and cumulants).

Furthermore, Pawula's theorem [25,26] brings interesting consequences for the KME, Eq. (5): whenever any of the higher order JM vanish ($n\geq 3$), all JM $a_{n\geq 3}=0$ and the FPE becomes exact. For instance, let us assume a distribution $\sigma(x,t|x_0,t_0)$ exists and is symmetric around x_0. The odd JM shall vanish, and consequently the higher order momenta disappear, and the even ones will also disappear except for the second order one, and the KME reduces to the usual Fokker–Planck Equation (FPE).

Let us first contrast Pawula's theorem restriction on the JM with the old but important result for general Probability Distribution Functions (PDF). It is well known that for a given PDF, theorem II of Marcinkiewicz's 1939 paper (MT) [27] guarantees that a PDF (having all of its momenta bounded) can either have an infinite number of non-zero cumulants, or only the first two, being then a Gaussian distribution. In fact, the PDF can present many null cumulants of order equal to or larger than 3, and still be a non-Gaussian distribution.

As an example, let us consider again the biased Poisson distribution described by the generating function $G(k)$ (all cumulants are $\langle x^n\rangle_c = \mu/2$):

$$G(k) = \left\langle e^{ikx}\right\rangle = e^{\frac{\mu}{2}\left(e^{ik}-1\right)}. \tag{10}$$

It is also well know that whenever $G(k)$ is a generating function, so are $G(-k)$ and $\phi(k) = G(k)\,G(-k)$. Thus we can obtain the generating function for an unbiased Poisson process

$$\phi(k) = e^{\frac{\mu}{2}\left(e^{ik}-1\right)}e^{\frac{\mu}{2}\left(e^{-ik}-1\right)} = e^{\mu(\cos(k)-1)}. \tag{11}$$

The generating function above yields the cumulants:

$$\left\langle x^{2n}\right\rangle_c = \mu, \tag{12}$$

$$\left\langle x^{2n+1}\right\rangle_c = 0. \tag{13}$$

As we can see, all the even ones are finite whereas the odd ones vanish. We shall return to this in the following. This implies that all odd averages are also zero

$$\left\langle x^{2n+1}\right\rangle = 0.$$

In the KME formalism, a process that generates stationary cumulants described by Eqs. (12) and (13) should either obey a FPE or a complete, and infinite, set of derivatives in the KME. However, neither seems like a good answer. In the following sections we shall develop a CKME for this problem.

4. Probability distribution for a generic dynamical system on *x*

We assume a bounded well-behaved dynamical system that can be characterized by a multi-variable x and its probability distribution function (PDF). All its moments and cumulants are assumed to converge. Indeed, this is the case for spatially restricted systems.

The PDF for this variable x, $p(x,t)$, shall observe the limitations according to MT [27], e.g., that its cumulants cannot be arbitrarily chosen: there is either an infinite number of non-zero cumulants or only two of them (Gaussian distribution).

The behavior of the dynamical system is expressed by $x(t) \equiv x(t;x_0,t_0)$, where the initial conditions are given by (x_0,t_0). We shall assume that the time evolution of $x(t)$ is well behaved enough so that the Laplace transform $\tilde{x}(s)$ exists, and is well behaved. The instantaneous probability distribution for the system is given by

$$p(x,t) = \left\langle \delta(x - x(t;x_0,t_0))\right\rangle, \tag{14}$$

where the average $\langle\rangle$ is taken over the initial conditions and over any other external factors that might influence the dynamics. For instance, in the case of a Langevin dynamics, this would correspond to taking the average over the noise.

It is straightforward to write the expression above for multi-points:

$$p(x_2,t_1;x_1,t_1) = \left\langle \delta(x_2 - x(t_2;x_0))\,\delta(x_1 - x(t_1;x_0))\right\rangle \tag{15}$$

hence the transition probability

$$p(x_2,t_1 \mid x_1,t_1) \equiv \frac{p(x_2,t_1;x_1,t_1)}{p(x_1,t_1)}. \tag{16}$$

By treating the expressions above in the same way as in Refs. [15–21], we obtain exactly the expressions below. Some details of the derivation are explicitly shown in Appendix A. For simplicity sake, we shall assume the initial time $t_0 = 0$ in what follows. The instantaneous PDF for the multivariable x can be written as (see Appendix A)

$$\begin{aligned} p(x,t) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}Q}{2\pi}\,\mathrm{e}^{\mathrm{i}Qx}\sum_{n=0}^{\infty}\frac{(-\mathrm{i}Q)^n}{n!}\left\langle x^n(t;x_0,0)\right\rangle \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}Q}{2\pi}\,\mathrm{e}^{\mathrm{i}Qx}\sum_{n=0}^{\infty}\frac{(-\mathrm{i}Q)^n}{n!}\int_{-\infty}^{\infty}\prod_{a=1}^{n}\frac{\mathrm{d}q_a}{2\pi}\,\mathrm{e}^{\sum\limits_{a=1}^{n}(\mathrm{i}q_a+\epsilon)(t-t_0)}\left\langle\prod_{a=1}^{n}\tilde{x}(\mathrm{i}q_a+\epsilon)\right\rangle, \end{aligned} \tag{17}$$

where we define the Laplace transform for x by

$$\tilde{x}(s) = \int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{-st}\,x(t;x_0,0). \tag{18}$$

Observe that the Markovian, or non-Markovian, character of the described dynamics is not yet determined above. These properties are implicit in the form of $\tilde{x}(s)$.

The equivalent multipoint distribution reads

$$\begin{aligned} p(x_1,t_1;\ldots;x_n,t_n) = &\int_{-\infty}^{\infty}\frac{\mathrm{d}Q_1}{2\pi}\cdots\frac{\mathrm{d}Q_n}{2\pi}\,\mathrm{e}^{\mathrm{i}(Q_1x_1+\cdots+Q_nx_n)}\prod_{j=1}^{n}\sum_{n_j=0}^{\infty}\frac{(-\mathrm{i}Q_j)^{n_j}}{n_j!} \\ &\times\prod_{j=1}^{n}\int_{-\infty}^{\infty}\prod_{a_j=1}^{n_j}\frac{\mathrm{d}q_{j,a_j}}{2\pi}\,\mathrm{e}^{\sum\limits_{a_j=1}^{n_j}(\mathrm{i}q_{j,a_j}+\epsilon)(t-t_0)}\left\langle\prod_{j=1}^{n}\prod_{a_j=1}^{n_j}\tilde{x}(\mathrm{i}q_{j,a_j}+\epsilon)\right\rangle. \end{aligned} \tag{19}$$

The forms above are general and exact and we can easily manipulate them. From them, we derive the time evolution equations for the instantaneous PDF.

5. Time evolution for the transition probability

We now focus on the class of Markovian dynamics, in other words, the description of the dynamics where

$$p(x_2, t_1 \mid x_1, t_1; x_0, t_0 < t_1) = p(x_2, t_1 \mid x_1, t_1). \tag{20}$$

A master equation is the evolution equation for the Markovian transition probability $p(x_2, t_1 \mid x_1, t_1)$ [23,28,29]. It is usually presented as a time-evolution equation for the PDF itself since we can assume that we know exactly the initial PDF for the problem, $p(x_0, 0) = \delta(x_0 - x_{initial})$. In doing so, we get to know $p(x, t) \equiv \int \mathrm{d}x_0\, p(x, t|x_0, t_0)\, p(x_0, t_0)$.

We can derive an exact equation evolution equation for $p(x, t|x_0, t_0)$ that is equivalent to the KME when the system is Markovian. Let us write the Kolmogorov equation

$$p(x, t+\tau|x_0, t_0) = \int \mathrm{d}x'\, p(x, t+\tau|x', t)\, p(x', t|x_0, t_0). \tag{21}$$

Now we take the derivative of $p(x, t+\tau|x', t)$ with respect to τ. Since the system is Markovian we do not need the information for the times before t. We shall take as initial coordinates the n-tuple (x', t). We can write $x \equiv x(t+\tau; x', t)$ and the transition probability then reads:

$$p(x, t+\tau \mid x', t) = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q}{2\pi}\, \mathrm{e}^{\mathrm{i}Qx} \sum_{n=0}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \left\langle \prod_{a=1}^{n} \tilde{x}(\mathrm{i}q_a+\epsilon) \right\rangle. \tag{22}$$

The generating function $\mathcal{G}(Q, \tau)$ (the Fourier transform of $p(x, t+\tau \mid x', t)$) and the cumulant generating function $\ln \mathcal{G}(Q, \tau)$ can be defined as

$$p(x, t+\tau \mid x', t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}Q}{2\pi}\, \mathrm{e}^{\mathrm{i}Qx}\, \mathcal{G}(Q, \tau), \tag{23}$$

$$\mathcal{G}(Q, \tau) = 1 + \lim_{\epsilon\to 0^+} \sum_{n=1}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \left\langle \prod_{a=1}^{n} \tilde{x}(\mathrm{i}q_a+\epsilon) \right\rangle, \tag{24}$$

$$\ln \mathcal{G}(Q, \tau) = \lim_{\epsilon\to 0^+} \sum_{n=1}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \left\langle \prod_{a=1}^{n} \tilde{x}(\mathrm{i}q_a+\epsilon) \right\rangle_c. \tag{25}$$

Let us derive a more specific equation for a 1D particle. We then define $x \equiv (y, v)$ and write the transition probability as

$$\begin{aligned} p(y, v, t+\tau \mid y', v', t) = {} & \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q}{2\pi}\, \mathrm{e}^{\mathrm{i}Qy} \sum_{n=1}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \int_{-\infty}^{\infty} \frac{\mathrm{d}P}{2\pi}\, \mathrm{e}^{\mathrm{i}Pv} \sum_{m=1}^{\infty} \frac{(-\mathrm{i}P)^m}{m!} \\ & \times \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \int_{-\infty}^{\infty} \prod_{b=1}^{m} \frac{\mathrm{d}p_m}{2\pi}\, \mathrm{e}^{\sum_{b=1}^{m}(\mathrm{i}p_b+\epsilon)\tau} \left\langle \prod_{a=1}^{n} \tilde{y}(\mathrm{i}q_a+\epsilon) \prod_{b=1}^{m} \tilde{v}(\mathrm{i}p_b+\epsilon) \right\rangle, \end{aligned} \tag{26}$$

and the corresponding cumulant generating function as

$$\begin{aligned} \ln \mathcal{G}(Q, P, \tau) = {} & \lim_{\epsilon\to 0^+} \sum_{n,m=0,\ n+m\geq 1}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \frac{(-\mathrm{i}P)^m}{m!} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \\ & \times \int_{-\infty}^{\infty} \prod_{b=1}^{m} \frac{\mathrm{d}p_m}{2\pi}\, \mathrm{e}^{\sum_{b=1}^{m}(\mathrm{i}p_b+\epsilon)\tau} \left\langle \prod_{a=1}^{n} \tilde{y}(\mathrm{i}q_a+\epsilon) \prod_{b=1}^{m} \tilde{v}(\mathrm{i}p_b+\epsilon) \right\rangle_c. \end{aligned} \tag{27}$$

We assume that all cumulants converge above.

From the cumulant generating function we can derive the CKME. Let us start from the general form for the probability distribution as a function of the cumulants $I_{y^n v^m}(y', v', t, \tau)$:

$$p(y, v, t+\tau \mid y', v', t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}Q}{2\pi}\, \mathrm{e}^{\mathrm{i}Qy} \int_{-\infty}^{\infty} \frac{\mathrm{d}P}{2\pi}\, \mathrm{e}^{\mathrm{i}Pv} \exp\left\{ \sum_{n+m\geq 1}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \frac{(-\mathrm{i}P)^m}{m!} I_{y^n v^m}(y', v', t, \tau) \right\}. \tag{28}$$

where $I_{y^n v^m}(y', v', t, \tau)$ are

$$\begin{aligned} I_{y^n v^m}(y', v', t, \tau) = {} & \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \int_{-\infty}^{\infty} \prod_{b=1}^{m} \frac{\mathrm{d}p_m}{2\pi}\, \mathrm{e}^{\sum_{b=1}^{m}(\mathrm{i}p_b+\epsilon)\tau} \\ & \times \left\langle \prod_{a=1}^{n} \tilde{y}(\mathrm{i}q_a+\epsilon) \prod_{b=1}^{m} \tilde{v}(\mathrm{i}p_b+\epsilon) \right\rangle_c. \end{aligned} \tag{29}$$

We derive Eq. (28) with respect to τ, which corresponds to deriving the cumulants $I_{y^n v^m}(y', v', t, \tau)$:

$$\begin{aligned}
\partial_\tau p(y, v, t+\tau \mid y', v', t) &= \int_{-\infty}^{\infty} \frac{dQ}{2\pi} e^{iQx} \int_{-\infty}^{\infty} \frac{dP}{2\pi} e^{iPy} \left\{ \sum_{n+m\geq 1}^{\infty} \frac{(-iQ)^n}{n!} \frac{(-iP)^m}{m!} I'_{y^n v^m}(y', v', t, \tau) \right\} \\
&\quad \times \exp\left\{ \sum_{n+m\geq 1}^{\infty} \frac{(-iQ)^n}{n!} \frac{(-iP)^m}{m!} I_{y^n v^m}(y', v', t, \tau) \right\} \\
&= \sum_{n+m\geq 1}^{\infty} \frac{(-1)^{n+m}}{n!m!} \frac{\partial^{n+m}}{\partial y^n \partial v^m} I'_{y^n v^m}(y', v', t, \tau) \int_{-\infty}^{\infty} \frac{dQ}{2\pi} e^{iQx} \int_{-\infty}^{\infty} \frac{dP}{2\pi} e^{iPy} \\
&\quad \times \exp\left\{ \sum_{n+m\geq 1}^{\infty} \frac{(-iQ)^n}{n!} \frac{(-iP)^m}{m!} I_{y^n v^m}(y', v', t, \tau) \right\} \\
&= \sum_{n+m\geq 1}^{\infty} \frac{(-1)^{n+m}}{n!\,m!} \frac{\partial^{n+m}}{\partial y^n \partial v^m} \left(I'_{y^n v^m}(y', v', t, \tau)\, p(y, v, t+\tau \mid y', v', t) \right).
\end{aligned} \tag{30}$$

We can define the Cumulant Jump-Moments (CJM) $a_{(n,m)}(y', v')$ as follows:

$$\begin{aligned}
a_{(n,m)}(y', v') &= \lim_{\tau\to 0^+} \partial_\tau I_{y^n v^m}(y', v', t, \tau) \\
&= \lim_{\tau\to 0^+} \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{dq_a}{2\pi} e^{\sum_{a=1}^{n}(iq_a+\epsilon)\tau} \int_{-\infty}^{\infty} \prod_{b=1}^{m} \frac{dp_m}{2\pi} e^{\sum_{b=1}^{m}(ip_b+\epsilon)\tau} \\
&\quad \times \left(\sum_{a=1}^{n} (iq_a+\epsilon) + \sum_{b=1}^{m} (ip_b+\epsilon) \right) \left\langle \prod_{a=1}^{n} \tilde{y}(iq_a+\epsilon) \prod_{b=1}^{m} \tilde{v}(ip_b+\epsilon) \right\rangle_c .
\end{aligned} \tag{31}$$

The similarities between the CJM above and the usual jump-moments are discussed in Appendix C. The Cumulant Kramers–Moyal Equation (CKME) is obtained by taking the limit $\tau \to 0$ of Eq. (30):

$$\partial_t p(y, v, t \mid y_0, v_0, t_0) = \sum_{n,m=0,\ n+m\geq 1}^{\infty} \frac{(-1)^{n+m}}{n!\,m!} \frac{\partial^{n+m}}{\partial y^n \partial v^m} \left[a_{(n,m)}(y, v)\ \ p(y, v, t \mid y_0, v_0, t_0) \right]. \tag{32}$$

In Appendix C we show that there is a simpler form for the jump-moment, namely:

$$a_{(n)}(x) = \lim_{\tau\to 0^+} \partial_\tau \left\langle x^n(\tau) \right\rangle_c , \tag{33}$$

which for a 1D problem can be written as

$$a_{(n,m)}(y, v) = \lim_{\tau\to 0^+} \partial_\tau \left\langle y^n(\tau)\, v^n(\tau) \right\rangle_c . \tag{34}$$

In Appendix C we compare the jump-moments above with the basic jump-moments for the KME [23].

Eq. (32) is a new proposal to treat processes that involve jumpy noise, such as Poisson. Other approaches involve complicated jump Kramers-like equations [30] such as

$$\partial_t p(y, v, t) = \left[-v\frac{\partial}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\gamma\, v + k\, y}{m} \right) \right] p(y, v, t) + \lambda \int d\Phi\, f(\Phi) \left[p\left(y, v - \frac{\Phi}{m}, t \right) - p\left(y, v + \frac{\Phi}{m}, t \right) \right]. \tag{35}$$

On the other hand, the CKME is simply a differential equation and can be dealt easier. In the next section we shall try to explain the main differences between the CKME and the KME approaches.

6. Comparing the CKME and the KME

An important point to stress out is that Eq. (32) involves no approximations whatsoever, thus no ambiguity arises from its derivation, distinctively from the KME which has presented some inconsistencies whenever distinct derivations were compared (see remark on Ref. [23]).

As mentioned earlier, the cumulant jump-moments of the CKME coincide with the KME ones up to second order, but no further. Further verification will be provided for a Poisson noise model that shows the CKME to furnish the correct stationary equilibrium averages.

An important distinction between the CKME and the KME consist on the nature of the coefficients, the cumulants for the CKME and the JM for the KME. We stress that the latter are formally calculated as averages of displacements starting from

(y, v), Eq. (6), while the former are the instantaneous averages for the dynamical variables, Eq. (34). The CKME approach might be very appropriate for problems where the dynamical averages are slowly changing in time.

Furthermore, the method used to derive Eq. (32) from Eq. (22) is not unique. In fact, we can decompose the average $A = \left\langle \prod_{a=1}^{n} \tilde{x}(iq_a + \epsilon) \right\rangle$ in the integrand of Eq. (22) into its cumulants components and, by rearranging the terms, we can obtain Eq. (31).[1]

The structure of the CKME is similar to the KME but with a fundamental difference: the CJM $a_{(n,m)}$ are derivatives of cumulants and are not in central moments format [31], like the JM for the KME. Indeed, the JM for the KME obey Pawula's theorem [25,26] (PT), while the CKME ones do not. It means that whenever any JM of order higher than 2 vanishes, all JM's of order higher than 2 also vanish and the KME reduces to the FPE. In other words, there are only two possibilities for the KME formalism: either it reduces to the FPE (a second order differential equation), or all infinite derivatives must be present in the KME.

On the other hand, the CKME is also an exact master-equation and the present formalism has some similarities with the one used to obtain the master-equation on the literature (see for instance Eqs. (3.29) and (3.30) of Hänggi's [32]).[2]

However, the properties of the CKME are related to the cumulant properties of a PDF for the variable x. It is known that for a given PDF, theorem II of Marcinkiewicz 1939 paper [27] (MT) guarantees that the PDF (having all of its momenta bounded) can either present an infinite number of non-zero cumulants (but not necessarily all of them, as we saw in Section 3 for case of the unbiased Poisson distribution where the odd cumulants were all zero), or only the first two of them, i.e., it has to be a Gaussian distribution. As stated a few years ago by Hanggi and Talkner [33], the MT cumulant limiting property can be seen as a weak form of PT since whenever the cumulants above a certain finite order vanish the generating function becomes Gaussian.

Let us go back to the Poisson problem. The biased Poisson model of Ref. [18] can be slightly modified as

$$p(\phi) = \frac{1}{2\phi_0} \, e^{-\frac{|\phi|}{\phi_0}}, \tag{36}$$

where the averages of ϕ are

$$\overline{\phi^{2n}} = (2n)! \, \phi_0^{2n}, \tag{37}$$

$$\overline{\phi^{2n+1}} = 0, \tag{38}$$

leading to non-zero even order cumulants, and null odd order ones. The model in Ref. [18], with the symmetrization above, is thus a good realization of the unbiased Poisson process in a dynamical setting. We shall see more details about this process in the following.

For a Brownian particle under additive Gaussian noise and a harmonic potential (as shall also be seen in the following) there is no difference between our results and the KME ones. For Gaussian noise, even for non-linear models, the character of the PDF is Gaussian and the CKME becomes the Fokker–Planck Equation (FPE). However, things get more interesting when the noise is no longer Gaussian and linear. When the higher order cumulants can couple with the non-linearity of the model [18]. In order to explore these characteristics of the CKME, we shall illustrate two paradigmatic linear cases that can be exactly treated by the CKME: the Gaussian Markovian model and the Poisson Markovian model.

In next sections we shall derive CKME for a few models and compare them with the FPE and KME approaches.

7. Gaussian Markovian model and Kramers equation

The Gaussian white noise (GWN) is a paradigmatic way to represent the coupling of a system with an equilibrium thermal reservoir. The long time state for a well behaved system coupled to GWN tends to the well known Boltzmann–Gibbs equilibrium distribution.

For the GWN the cumulants are given as

$$\langle \xi(t_1) \rangle_c = 0 \tag{39}$$

$$\langle \xi(t_1)\, \xi(t_2) \rangle_c = 2\, \gamma\, T\, \delta(t_1 - t_2), \tag{40}$$

where the higher order cumulants are zero. In this case the Langevin equation for the harmonic potential case is simply

$$m\, \ddot{x} = -\gamma\, \dot{x} - k\, x + \xi. \tag{41}$$

[1] This can be seen by using some symmetry arguments concerning the q-moments that will multiply A due to the time-derivation and by singling out one of these moments, let us rename it q_0, and by keeping the cumulant it belongs to: $\propto \int \sum (iq_0 + \epsilon) \left\langle \tilde{x}(iq_0 + \epsilon) \ldots \right\rangle_c \left\langle \tilde{x}(iq_i + \epsilon) \ldots \right\rangle_c \ldots$, which leads to $\propto \int \sum (iq_0 + \epsilon) \left\langle \tilde{x}(iq_0 + \epsilon) \ldots \right\rangle_c \langle \cdots \rangle$ while adding all other products of cumulants in order to reconstruct an average. By carefully rearranging the terms we can arrive at the same result, namely Eq. (32), via Eq. (31).

[2] In the present work we assume a general form for the dynamical system. Only when making direct application in the context of Langevin dynamics we use an additive noise but the structure of the jump-moments is still quite distinct from that of the KME.

The Laplace transform of the dynamics gives us ($x \equiv (y, v)$ and the initial conditions are (y', v'))

$$\tilde{y}(s) = \frac{\left[m\,(v' + s\,y') + \gamma\,y'\right]}{R(s)} + \frac{\tilde{\xi}(s)}{R(s)}, \tag{42}$$

$$\tilde{v}(s) = -y' + \frac{s\left[m\,(v' + s\,y') + \gamma\,y'\right]}{R(s)} + \frac{s\,\tilde{\xi}(s)}{R(s)}, \tag{43}$$

where $R(s) = (ms^2 + \gamma\,s + k)$. Observe that $a_{(1,0)}$ is a drift in y-space and $a_{(0,1)}$ is an acceleration (a drift in v-space). The JM $a_{(0,2)}$ is a thermal diffusion term in v-space (see Appendix C).

The cumulants and jump moments are obtained in Appendix D. We only obtain non-zero jump moments up to second order, all others are zero due to the Gaussian property of the noise. They are:

$$a_{(1,0)} = v', \tag{44}$$

$$a_{(0,1)} = -\frac{\gamma\,v' + y'k}{m}, \tag{45}$$

$$a_{(0,2)} = \frac{2\,\gamma\,T}{m^2}. \tag{46}$$

These CJM lead us to the CKME which is Kramers equation ($V'(y) = -k\,y$):

$$\partial_t p(y, v, t) = -v\frac{\partial}{\partial y}p(y, v, t) + \frac{\partial}{\partial v}\left[\left(\frac{\gamma\,v + k\,y}{m}\right)p(y, v, t)\right] + \frac{\gamma\,T}{m^2}\frac{\partial^2}{\partial v^2}p(y, v, t). \tag{47}$$

The departure, in the following, from Gaussianity for Poisson additive noise makes the analysis more interesting.

8. Biased and unbiased Poisson noise CKME

Following the same basic calculations of Ref. [20], we can evaluate the form for the jump moments, where the cumulants of the noise are

$$\left\langle \tilde{\xi}(s_1)\ldots\tilde{\xi}(s_n)\right\rangle_c = \frac{\lambda\,\overline{\phi^n}}{\sum\limits_{i=1}^{n} s_i}. \tag{48}$$

Using the same model as in Ref. [20], we can calculate the jump moments exactly. They are obtained in Appendix E. The existence and origin of non equilibrium and athermal reservoirs, such as the one described by Eq. (48), have been the subject of many studies. A very interesting proposal appeared in the recent Kanazawa et al. paper [34], where the authors show a physical (granular) realization of such reservoirs.

8.1. Biased Poisson noise

The CKME for the biased distribution is:

$$\partial_t p = -\frac{\partial}{\partial y}(v\,p) + \frac{\partial}{\partial v}\left(\left[\frac{\gamma v + ky}{m} - \frac{\lambda\,\overline{\phi}}{m}\right]p\right) + \sum_{s=2}^{\infty}\frac{(-1)^s\,\lambda\,\overline{\phi^s}}{m^s}\frac{\partial^s}{\partial v^s}\,p. \tag{49}$$

Although there is an infinite number of derivatives in Eq. (49) it still is of the KME form. Below we study some of the equilibrium properties of the averages that can be extracted from Eq. (49).

The stationary distribution obeys

$$0 = -\frac{\partial}{\partial y}(v\,p_{ss}) + \frac{\partial}{\partial v}\left(\left[\frac{\gamma v + ky}{m} - \frac{\lambda\,\overline{\phi}}{m}\right]p_{ss}\right) + \sum_{s=2}^{\infty}\frac{(-1)^s\,\lambda\,\overline{\phi^s}}{m^s}\frac{\partial^s}{\partial v^s}\,p_{ss}.$$

From above it is straightforward to derive the stationary momenta of (y, v). Among them an interesting identity results (multiply the equation above by y^{n+1} and integrate by parts)

$$\left\langle y^n v\right\rangle = 0 \Rightarrow \langle F(y)\,v\rangle = 0, \tag{50}$$

which implies that the velocity does not couple to functions of the position, in special with the potential forces acting on the particle.

Collecting the results for the first non-zero momenta $\langle y^r v^s \rangle$, we obtain ($\frac{\lambda \overline{\phi}^2}{\gamma} = T$) for the stationary averages up to order 3 on the dynamical variables and for the total energy:

$$\langle y \rangle = \frac{\lambda \overline{\phi}}{k},$$
$$\langle v^2 \rangle = \frac{T}{m},$$
$$\langle y^2 \rangle = \frac{T}{k} + \frac{\lambda^2 \overline{\phi}^2}{k^2},$$
$$\langle y v^2 \rangle = \frac{\left([km + 2\gamma^2]\lambda + 2k\gamma\right)}{\lambda \left(km + 2\gamma^2\right)} \left(\frac{T}{m} \frac{\lambda \overline{\phi}}{k}\right)$$
$$\langle y^3 \rangle = \frac{4\gamma}{\left(2\gamma^2 + km\right)} \frac{T \overline{\phi}}{k},$$
$$\langle v^3 \rangle = \frac{4\gamma^2}{m\left(2\gamma^2 + km\right)} \frac{T \overline{\phi}}{m},$$
$$\begin{aligned}\langle E \rangle &= \frac{1}{2} m \langle v^2 \rangle + \frac{1}{2} k \langle y^2 \rangle \\ &= T + \frac{1}{2} \frac{\lambda^2 \overline{\phi}^2}{k}.\end{aligned}$$

These same moments can be obtained directly from the dynamics [20]. It verifies the exactness of the CKME.

8.2. Unbiased Poisson noise

Similarly, the CKME for the unbiased equation is (slightly distinct from the biased case):

$$\partial_t p = -\frac{\partial}{\partial y}(v\, p) + \frac{\partial}{\partial v}\left(\left[\frac{\gamma v + k y}{m}\right] p\right) + \sum_{s=1}^{\infty} \frac{\lambda \overline{\phi^{2s}}}{m^{2s}} \frac{\partial^{2s}}{\partial v^{2s}}. \tag{51}$$

Only the even powers of the velocity derivatives remain at orders above 2, which makes this equation deviate from the usual KME form. The odd moments then vanish since now the PDF becomes symmetric in y and v. Certainly, the CKME does not obey Pawula's theorem and Eq. (51) is unique.

We can collect all the exact stationary averages (up to order 4 below), the non-zero ones being (the energy average holds: $\langle E \rangle = T$ since the stretching energy disappears due to the symmetry of the noise term)

$$\langle v^2 \rangle = \frac{T}{m},$$
$$\langle y^2 \rangle = \frac{T}{k},$$
$$\langle y^4 \rangle = \frac{3T^2\left(6\gamma k + 4\lambda km + 3\lambda\gamma^2\right)}{\lambda\left(4km + 3\gamma^2\right)k^2},$$
$$\langle y^2 v^2 \rangle = \frac{T^2\left(6\gamma k + 4\lambda km + 3\lambda\gamma^2\right)}{\lambda\left(4km + 3\gamma^2\right)km},$$
$$\langle y v^3 \rangle = \frac{6T^2\gamma^2}{m^2\lambda\left(4km + 3\gamma^2\right)},$$
$$\langle v^4 \rangle = \frac{3T^2\left(6\gamma km + 4m^2\lambda k + 3m\lambda\gamma^2 + 6\gamma^3\right)}{m^3\lambda\left(4km + 3\gamma^2\right)}.$$

In the values for the averages above we see the effect of the fourth cumulant via some of the terms dependent on the Poisson rate. In special, the stationary cumulant term $\langle y v^3 \rangle = \langle y v^3 \rangle_c$ should be zero in the context of the Fokker–Planck approximation. Via the CKME we obtain the correct exact value given above, which can also be derived using the techniques of Ref. [20].

The CKME we obtain, Eq. (51), leads to the exact forms for the averages of the dynamical variables, unlike the FPE in this case.

9. Unbiased Poisson: a simple model

The present CKME can be applied in many settings that are not necessarily associated with particle dynamics. Let us propose a 1D continuous model that starts at $x_0 = 0$ and gives the position x at time t as

$$x(t) = \sum_{i=1}^{\infty} \phi(t_i)\, \Theta(t - t_i), \tag{52}$$

where the times t_i are distributed with a Poisson rate λ, and the intensity of the Poisson kick ϕ follows the distribution

$$p(\phi) = \frac{e^{-\left|\frac{\phi}{\phi_0}\right|}}{2\,\phi_0}, \tag{53}$$

and finally Θ is the Heaviside function. The parameter ϕ_0 is the typical step length. The model is Markovian since its jump probabilities do not depend on the past trajectory.

The characteristic function for the probability $p(x, t)$ can be easily calculated since the probabilities that a number n of events happening in the interval $[0, \lambda\, t]$ is

$$p(n, t) = \frac{e^{-\lambda t}\,(\lambda\, t)^n}{n!}. \tag{54}$$

Let us define the position X as the sum over variables ϕ_i distributed on times t_i according to the Poisson rate above:

$$X_n(t) = \sum_{i}^{n} \phi_i. \tag{55}$$

The instantaneous generating function reads:

$$\mathcal{G}(k, t) = \sum_{n=0}^{\infty} p(n, t) \int_{-\infty}^{\infty} \prod_{i=1}^{n} d\phi_i \prod_{i=1}^{n} p(\phi_i)\, e^{-ikX_n}. \tag{56}$$

Replacing the above distributions we get

$$\begin{aligned}
\mathcal{G}(k, t) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}\,(\lambda\, t)^n}{n!} \int_{-\infty}^{\infty} \prod_{i=1}^{n} d\phi_i \prod_{i=1}^{n} \frac{e^{-\left|\frac{\phi_i}{\phi_0}\right|}}{2\,\phi_0} \prod_{i=1}^{n} e^{-ik\phi_i} \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}\,(\lambda\, t)^n}{n!} \left[\int_{-\infty}^{\infty} d\phi\, \frac{e^{-\left|\frac{\phi}{\phi_0}\right|}}{2\,\phi_0}\, e^{-ik\phi} \right]^n \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}\,(\lambda\, t)^n}{n!} \left[\frac{1}{1 + k^2\,\phi_0^2} \right]^n \\
&= e^{-\frac{k^2\,\phi_0^2\,\lambda t}{1+k^2\,\phi_0^2}}.
\end{aligned} \tag{57}$$

The instantaneous cumulant generating function is (converging when expanded for $|k| \le \frac{1}{\phi_0}$)

$$\ln \mathcal{G}(k, t) = -\frac{k^2\,\phi_0^2\,\lambda t}{1 + k^2\,\phi_0^2} = \sum_{n=1}^{\infty} \frac{(-k^2)^n}{(2n)!}\,(2n)!\,\phi_0^{2n}\,\lambda t. \tag{58}$$

The instantaneous cumulants can be calculated:

$$\left\langle x^{2n} \right\rangle_c = (2n)!\,\phi_0^{2n}\,\lambda t, \tag{59}$$

$$\left\langle x^{2n+1} \right\rangle_c = 0. \tag{60}$$

The model above shows the same cumulant structure than the unbiased Poisson Langevin problem of the previous section.

We can construct the CKME for the model since

$$a_{2n} = \partial_t \left\langle x^{2n} \right\rangle_c = (2n)!\,\phi_0^{2n}\,\lambda,$$

and the odd ordered CJM are null. We have the CKME:

$$\partial_t p(x, t) = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{(2n)!} \frac{\partial^{2n}}{\partial x^{2n}} \left[(2n)!\,\phi_0^{2n}\,\lambda\, p(x, t) \right] = \lambda \sum_{n=1}^{\infty} \phi_0^{2n} \frac{\partial^{2n}}{\partial x^{2n}}\, p(x, t). \tag{61}$$

In the limit $\phi_0 \to 0,\ \lambda \to \infty$ with $\lambda\,\phi_0^2 = D$ finite, the CKME becomes the expected simple diffusion equation. Observe that higher order cumulants, Eq. (59), are proportional to $\phi_0^{2n}\lambda = \phi_0^{2(n-1)}D \to 0$ as we take the high rate, low intensity limit. In fact, that limit recovers the Gaussian case and the CKME becomes the FPE.

From the equation above we obtain after integrating by parts:

$$\dot{\left\langle x^2\right\rangle} = 2\,\lambda\,\phi_0^2 \Rightarrow \left\langle x^2\right\rangle = 2\,\lambda\,t\,\phi_0^2 = \left\langle x^2\right\rangle_c , \tag{62}$$

and

$$\dot{\left\langle x^4\right\rangle} = 24\,\lambda^2\,\phi_0^4\,t + 4!\,\lambda\,\phi_0^4 \Rightarrow 12\,\lambda^2\,\phi_0^4\,t^2 + 4!\,\lambda\,\phi_0^4\,t = 3\left\langle x^2\right\rangle_c^2 + \left\langle x^4\right\rangle_c , \tag{63}$$

as expected. All other averages can be obtained exactly from the CKME.

The CKME above is certainly not a FPE neither a KME (odd order derivatives are missing) and still it is exact.

10. Conclusions

We start with a generic, well behaved, Markovian system and derive the transition probability in a general way. From it, we obtain an exact evolution equation for the instantaneous probability distribution similar to the Kramers–Moyal Equation (KME), where the jump-moments are based on cumulants instead of central moments as is usual, the cumulant Kramers–Moyal Equation (CKME). By construction, the positivity of the PDF is guaranteed and the CKME preserves the normalization of probabilities.

One important difference is that the CKME obeys a weaker version of Pawula's theorem [25,26] (PT), namely Marcinkiewicz theorem [27] (MT) that states that for a given probability distribution either it has an infinite number of non-zero cumulants or the distribution is Gaussian. For practical applications, whenever the higher order cumulants become negligible, the distribution becomes Gaussian and the CKME reduces to the usual Fokker–Planck equation.

We apply these equations to some well known models, such as Gaussian and Poisson Langevin systems. For the Gaussian case we obtain the expected Kramers equation in an exact manner. For the Poisson noise case the results were more interesting since we study both the biased and the unbiased Poisson noise in order to derive the CKME. The biased Poisson model leads to an equation of the KME form. However, for the unbiased case we obtain a CKME that is distinct from the KME due to the presence of even velocity derivatives, and only the even ones. From there we can check all the equilibrium momenta and compare them with the ones obtained from other methods. We found them to be identical as expected.

A simple non-Langevin Markovian unbiased Poisson model is also exploited and the corresponding CKME is the exact equation for the time evolution of its PDF. It is neither of the FPE nor the KME form.

The action of external interactions (forces) could be easily encapsulated by the jump-moments $a_{(n,m)}(y, v)$. Thus, one could integrate the effect of these external influences and account for their effect into the macroscopic quantities, such as external work, obtained via averages with the probability distribution. But we shall not go any further along these lines presently.

Summing it all up, the CKME allows us to obtain all the exact stationary cumulants for many models, including the unbiased Poisson dynamics.

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Appendix A. Laplace expressions

Some of the more non-standard calculations in this work are derived below. The method has been used in Refs. [15–22]. The instantaneous PDF is:

$$\begin{aligned} p(x,t) &= \left\langle \delta(x - x(t;x_0,0))\right\rangle , \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}Q}{2\pi} \left\langle \mathrm{e}^{\mathrm{i}Q\,(x - x(t;x_0,0))}\right\rangle \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}Q}{2\pi}\, \mathrm{e}^{\mathrm{i}Qx} \sum_{n=0}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \left\langle x^n(t;x_0,t_0)\right\rangle \\ &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q}{2\pi}\, \mathrm{e}^{\mathrm{i}Qx} \sum_{n=0}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \int_0^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \frac{\mathrm{d}q_1}{2\pi}\, \mathrm{e}^{(\mathrm{i}q_1+\epsilon)(t-t_1)} \\ &\quad \times \cdots \times \int_0^{\infty} \mathrm{d}t_n \int_{-\infty}^{\infty} \frac{\mathrm{d}q_n}{2\pi}\, \mathrm{e}^{(\mathrm{i}q_n+\epsilon)(t-t_n)} \left\langle x(t_1;x_0,0)\ldots x(t_n;x_0,0)\right\rangle , \end{aligned} \tag{A.1}$$

and now we integrate over the t_i:

$$p(x,t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}Q}{2\pi}\, \mathrm{e}^{\mathrm{i}Qx} \sum_{n=0}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)t} \left\langle \prod_{a=1}^{n} \tilde{x}(\mathrm{i}q_a+\epsilon) \right\rangle, \tag{A.2}$$

where we define the Laplace transform for x by

$$\tilde{x}(s) = \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-st}\, x(t; x_0, 0). \tag{A.3}$$

Appendix B. The CJM

The instantaneous cumulant expression for a variable can be obtained from Eq. (A.2) yielding the cumulant generating function for the multi-variable x (here we assume (x', t) to be the initial conditions)

$$\ln \mathcal{G}(Q,\tau) = \sum_{n=1}^{\infty} \frac{(-\mathrm{i}Q)^n}{n!} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \left\langle \prod_{a=1}^{n} \tilde{x}(\mathrm{i}q_a+\epsilon) \right\rangle_c ,$$

and the nth instantaneous cumulant $I_{x^n}(x', t, \tau) \equiv \langle x^n(t+\tau)\rangle_c$

$$I_{x^n}(x',t,\tau) = \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \left\langle \prod_{a=1}^{n} \tilde{x}(\mathrm{i}q_a+\epsilon) \right\rangle_c . \tag{B.1}$$

The CJM can be written in the equivalent forms:

$$a_n(x',t) = \lim_{\tau\to 0} \partial_\tau I_{x^n}(x',t,\tau), \tag{B.2}$$

$$= \lim_{\tau\to 0} \partial_t I_{x^n}(x',t,\tau), \tag{B.3}$$

$$= \lim_{\tau\to 0} \partial_\tau \left\langle x^n(t+\tau)\right\rangle_c , \tag{B.4}$$

$$= \lim_{\tau\to 0} \partial_t \left\langle x^n(t+\tau)\right\rangle_c , \tag{B.5}$$

$$= \partial_t \left\langle x^n(t)\right\rangle_c , \tag{B.6}$$

$$= \lim_{\tau\to 0} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi} \left(\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\right) \mathrm{e}^{\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \left\langle \prod_{a=1}^{n} \tilde{x}(\mathrm{i}q_a+\epsilon) \right\rangle_c . \tag{B.7}$$

Appendix C. The cumulant-jump moments and the Kramers–Moyal jump-moments

For the CKME we observe that the cumulant jump moment of Eq. (31) is exactly

$$a_{(n)}(x) = \lim_{\tau\to 0^+} \partial_\tau \left\langle x^n(t+\tau)\right\rangle_c . \tag{C.1}$$

Let us express the $n=2$ case:

$$a_{(n)}(x) = \lim_{\tau\to 0^+} \partial_\tau \left\langle (x(t+\tau) - \langle x(t+\tau)\rangle)^n \right\rangle$$

$$= \lim_{\tau\to 0^+} \int \mathrm{d}x\, \frac{p(x, t+\tau \mid x', t)}{\tau}\, (x - \langle x(t+\tau)\rangle)^n , \tag{C.2}$$

where we need to understand the role of the averages above.

Let us approximate the new position x as a function of the older position x' plus a generalized drift and a diffusion term η. It gives

$$x = x' + \bar{v}\,\tau + \eta, \tag{C.3}$$

where $\bar{v}$ is the generalized drift velocity and $\langle \eta^2\rangle = \sigma^2 = D\tau$ ($\langle\eta\rangle = 0$). The parameters above are given by the Fokker–Planck JM's where

$$a_1 \equiv \lim_{\tau\to 0^+} \int \mathrm{d}x\, \frac{p(x, t+\tau \mid x', t)}{\tau} \left(x - x'\right) = \bar{v},$$

and

$$a_2 \equiv \lim_{\tau\to 0^+} \int \mathrm{d}x \,\frac{p(x, t+\tau \mid x', t)}{\tau}\, \left(x - x'\right)^2 = D.$$

We take the derivative of the average of x which gives

$$\partial_\tau \langle x\rangle_c = \partial_\tau \langle x\rangle = \partial_\tau (x' + \bar{v}\tau) = \bar{v},$$

as expected. Thus, the average $\langle x(t+\tau)\rangle$ is well represented by $\langle x(t+\tau)\rangle_c = x' + \bar{v}\tau$.

Let us take a look at the second CJM. It is also correct since

$$\lim_{\tau\to 0^+} \partial_\tau \left\langle (x - \langle x(t+\tau)\rangle)^2\right\rangle = \lim_{\tau\to 0^+} \partial_\tau \left\langle (x - x' - \bar{v}\tau)^2\right\rangle = \lim_{\tau\to 0^+} \partial_\tau \langle \eta^2\rangle = D.$$

Consequently, the first two CJM coincide with the first two JM for the KME. However, that might not be true for the subsequent JM. For cumulant jump-moments of higher order, it is no longer possible to compare them directly with the jump moments of the KME.

Appendix D. The Gaussian jump moments

From Eqs. (42) and (43), we observe that the averages $\langle\tilde{y}\rangle$ and $\langle\tilde{v}\rangle$ shall involve the initial conditions but the variance, skewness and higher order cumulants shall only involve averages of the noise. In fact, for Gaussian noise only the variance has to be taken into account, all the higher order cumulants being zero.

For $n + m = 1$, only the terms from the initial conditions shall contribute since the average of the noise is zero. By replacing $\langle\tilde{y}\rangle$ in it and proceeding to calculate (the poles of $R(s)$ are located on the upper-half-plane)

$$\begin{aligned} a^{Gaussian}_{(1,0)}(y, v) &= \lim_{\tau\to 0^+}\lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{(\mathrm{i}q_a+\epsilon)\tau}\, (\mathrm{i}q_a+\epsilon)\, \frac{\left[m\,(v' + [\mathrm{i}q_a+\epsilon]\,y') + \gamma\, y'\right]}{m\,R(\mathrm{i}q_a+\epsilon)} \\ &= v'. \end{aligned} \tag{D.1}$$

The other order-one JM can also be obtained as easily.

For $n + m = 2$, only the noise averages will contribute and generate a JM of the form

$$\begin{aligned} a^{Gaussian}_{(n,m)}(y, v) = {}& \lim_{\tau\to 0^+}\lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \prod_{a=1}^{n} \frac{\mathrm{d}q_a}{2\pi}\, \mathrm{e}^{\sum\limits_{a=1}^{n}(\mathrm{i}q_a+\epsilon)\tau} \int_{-\infty}^{\infty} \prod_{b=1}^{m} \frac{\mathrm{d}p_m}{2\pi}\, \mathrm{e}^{\sum\limits_{b=1}^{m}(\mathrm{i}p_b+\epsilon)\tau} \\ & \times \left(\sum_{a=1}^{n}(\mathrm{i}q_a+\epsilon) + \sum_{b=1}^{m}(\mathrm{i}p_b+\epsilon)\right) \prod_{b=1}^{m}(\mathrm{i}p_b+\epsilon) \times \frac{\left\langle \prod\limits_{a=1}^{n} \tilde{\xi}(\mathrm{i}q_a+\epsilon)\, \prod\limits_{b=1}^{m} \tilde{\xi}(\mathrm{i}p_b+\epsilon)\right\rangle_c}{m^{n+m} \prod\limits_{a=1}^{n} R(\mathrm{i}q_a+\epsilon)\, \prod\limits_{b=1}^{m} R(\mathrm{i}p_b+\epsilon)}. \end{aligned} \tag{D.2}$$

We use the Laplace transform of the noise and can calculate the JM.

The only non-zero ones are given by:

$$a_{(1,0)} = v', \tag{D.3}$$

$$a_{(0,1)} = -\frac{\gamma\, v' + y'k}{m}, \tag{D.4}$$

$$a_{(0,2)} = \frac{2\,\gamma\, T}{m^2}. \tag{D.5}$$

The coefficients above, when used in Eq. (32), yield the usual Kramers equation [23].

Alternatively, one can calculate the $I_{y^n v^m}$ cumulants and take the time derivative at $\tau \to 0$. The complete form for the distribution cumulants for the Gaussian case are given below:

$$I_y = -\frac{\left(mv' + m\kappa_2 y' + \gamma y'\right)\mathrm{e}^{\kappa_2\tau}}{m\,(\kappa_1 - \kappa_2)} + \frac{\left(mv' + m\kappa_1 y' + \gamma y'\right)\mathrm{e}^{\kappa_1\tau}}{m\,(\kappa_1 - \kappa_2)}, \tag{D.6}$$

$$I_v = -\frac{\kappa_2\,(mv + m\kappa_2 y + \gamma y)\,\mathrm{e}^{\kappa_2\tau}}{m\,(\kappa_1 - \kappa_2)} + \frac{\kappa_1\,(mv + m\kappa_1 y + \gamma y)\,\mathrm{e}^{\kappa_1\tau}}{m\,(\kappa_1 - \kappa_2)} \tag{D.7}$$

and

$$I_{yy} = \frac{T}{k} + \frac{gT\left(\kappa_2 e^{2\kappa_1\tau}\kappa_1 + e^{2\kappa_1\tau}\kappa_2{}^2\right)}{m^2\left(\kappa_1-\kappa_2\right)^2\kappa_1\left(\kappa_1+\kappa_2\right)\kappa_2} + \frac{gT\left(-4\kappa_2 e^{\tau(\kappa_1+\kappa_2)}\kappa_1 + e^{2\kappa_2\tau}\kappa_1{}^2 + \kappa_1 e^{2\kappa_2\tau}\kappa_2\right)}{m^2\left(\kappa_1-\kappa_2\right)^2\kappa_1\left(\kappa_1+\kappa_2\right)\kappa_2}, \tag{D.8}$$

$$I_{yv} = \frac{\left(e^{2\kappa_1\tau} - 2\,e^{\tau(\kappa_1+\kappa_2)} + e^{2\kappa_2\tau}\right)gT}{m^2\left(\kappa_1-\kappa_2\right)^2} \tag{D.9}$$

$$I_{vv} = \frac{T}{m} + \frac{gT\left(e^{2\kappa_1\tau}\kappa_1{}^2 + \kappa_1 e^{2\kappa_1\tau}\kappa_2\right)}{m^2\left(\kappa_1-\kappa_2\right)^2\left(\kappa_1+\kappa_2\right)} + \frac{gT\left(-4\kappa_1 e^{\tau(\kappa_1+\kappa_2)}\kappa_2 + \kappa_2 e^{2\kappa_2\tau}\kappa_1 + e^{2\kappa_2\tau}\kappa_2{}^2\right)}{m^2\left(\kappa_1-\kappa_2\right)^2\left(\kappa_1+\kappa_2\right)}. \tag{D.10}$$

At the limit $\tau \to 0$ we obtain, by taking the derivative of the above cumulants, the non-zero jump-moments above, as we have done by the direct integration method.

Appendix E. The biased and unbiased Poisson jump moments

Let us once again remember that the averages $\langle\tilde{y}\rangle$ and $\langle\tilde{v}\rangle$ shall involve the initial conditions but the variance, skewness and higher order cumulants shall only involve averages of the noise.

The Laplace transform for the Poisson noise cumulants is given by [20]

$$\langle\tilde{\eta}(iq_1+\epsilon)\ldots\tilde{\eta}(iq_n+\epsilon)\rangle_c = \frac{\lambda\,\overline{\phi^n}}{\sum_{a=1}^{n}(iq_a+\epsilon)}. \tag{E.1}$$

The first order ones are almost the same as the Gaussian case, except for a contribution from the noise term in the biased Poisson case ($\bar{\phi} \neq 0$). The case for $n+m \geq 2$ only involve noise contributions as shown below:

$$\begin{aligned}
a^{Poisson}_{(n,m)}(y,v) &= \lim_{\tau\to0^+}\lim_{\epsilon\to0^+}\int_{-\infty}^{\infty}\prod_{a=1}^{n}\frac{dq_a}{2\pi}\,e^{\sum_{a=1}^{n}(iq_a+\epsilon)\tau}\int_{-\infty}^{\infty}\prod_{b=1}^{m}\frac{dp_m}{2\pi}\,e^{\sum_{b=1}^{m}(ip_b+\epsilon)\tau}\\
&\quad\times\left(\sum_{a=1}^{n}(iq_a+\epsilon)+\sum_{b=1}^{m}(ip_b+\epsilon)\right)\prod_{b=1}^{m}(ip_b+\epsilon)\times\frac{\left\langle\prod_{a=1}^{n}\tilde{\xi}(iq_a+\epsilon)\ \prod_{b=1}^{m}\tilde{\xi}(ip_b+\epsilon)\right\rangle_c}{m^{n+m}\ \prod_{a=1}^{n}R(iq_a+\epsilon)\ \prod_{b=1}^{m}R(ip_b+\epsilon)}\\
&= \lim_{\tau\to0^+}\lim_{\epsilon\to0^+}\int_{-\infty}^{\infty}\prod_{a=1}^{n}\frac{dq_a}{2\pi}\,e^{\sum_{a=1}^{n}(iq_a+\epsilon)\tau}\int_{-\infty}^{\infty}\prod_{b=1}^{m}\frac{dp_m}{2\pi}\,e^{\sum_{b=1}^{m}(ip_b+\epsilon)\tau}\\
&\quad\times\frac{\prod_{b=1}^{m}(ip_b+\epsilon)\,\lambda\,\overline{\phi^{n+m}}}{m^{n+m}\ \prod_{a=1}^{n}R(iq_a+\epsilon)\ \prod_{b=1}^{m}R(ip_b+\epsilon)}\\
&= \frac{\lambda\,\overline{\phi^{n+m}}}{m^{n+m}}\,I^n_{Py}\,I^m_{Pv}
\end{aligned} \tag{E.2}$$

where

$$\begin{aligned}
I_{Py} &= \lim_{\tau\to0^+}\lim_{\epsilon\to0^+}\int_{-\infty}^{\infty}\frac{dq_a}{2\pi}\,e^{(iq_a+\epsilon)\tau}\frac{1}{R(iq_a+\epsilon)},\\
&= 0,
\end{aligned} \tag{E.3}$$

$$\begin{aligned}
I_{Pv} &= \lim_{\tau\to0^+}\lim_{\epsilon\to0^+}\int_{-\infty}^{\infty}\frac{dp_b}{2\pi}\,e^{(ip_b+\epsilon)\tau}\frac{(ip_b+\epsilon)}{R(ip_b+\epsilon)},\\
&= 1.
\end{aligned} \tag{E.4}$$

From above we see that

$$a^P_{(n\geq1,m\geq1)}(y,v) = 0. \tag{E.5}$$

We assume that the intensity of the Poisson kick is exponentially distributed yielding for the biased case as in Ref. [20]

$$\overline{\phi^n} = (n)!\,\overline{\phi}^n.$$

For the unbiased case only the even $n = 2p$ averages survive.

The only non-zero jump moments are then ($n = 0,\ s \geq 2$)

$$a^{P}_{(0,s)}(y, v) = s!\,\frac{\lambda\,\overline{\phi}^{s}}{m^{s}}. \tag{E.6}$$

For the case $r + s = 1$ we have to be more careful and differentiate the biased and unbiased cases. For the biased case:

$$a^{PB}_{(1,0)}(y, v) = v, \tag{E.7}$$

$$a^{PB}_{(0,1)}(y, v) = -\frac{\gamma v + ky}{m} + \frac{\lambda\,\overline{\phi}}{m}. \tag{E.8}$$

We notice a stretching term due to the bias $\frac{\lambda\overline{\phi}}{m}$. It acts as a constant force upon the Brownian particle, as shown in Ref. [20]. For the unbiased case:

$$a^{PU}_{(1,0)}(y, v) = v, \tag{E.9}$$

$$a^{PU}_{(0,1)}(y, v) = -\frac{\gamma v + ky}{m}. \tag{E.10}$$

We can now construct the CKME and the conjugate operator for the Poisson cases. The conjugate stationary operator for the biased Poisson is

$$\mathcal{O}^{\dagger} = v\,\frac{\partial}{\partial y} - \left(\frac{\gamma v + ky}{m} - \frac{\lambda\,\overline{\phi}}{m}\right)\frac{\partial}{\partial v} + \sum_{s=2}^{\infty}\frac{\lambda\,\overline{\phi}^{s}}{m^{s}}\,\frac{\partial^{s}}{\partial v^{s}}, \tag{E.11}$$

while for the unbiased case we have

$$\mathcal{O}^{\dagger} = v\,\frac{\partial}{\partial y} - \left(\frac{\gamma v + ky}{m}\right)\frac{\partial}{\partial v} + \sum_{s=1}^{\infty}\frac{\lambda\,\overline{\phi}^{2s}}{m^{2s}}\,\frac{\partial^{2s}}{\partial v^{2s}}. \tag{E.12}$$

The stationary averages $\left\langle y^{n'} v^{m'}\right\rangle$ are obtained from hierarchically solving the equations

$$0 = \int \mathrm{d}y\,\mathrm{d}v\,p_{ss}\,\mathcal{O}^{\dagger} y^{n} v^{m}.$$

Notice that this is possible due to the confinement of the particle around the harmonic potential. It ensures the probabilities will not vanish as in the case of purely diffusing free-particle case.

References

[1] J.L. Lebowitz, Statistical mechanics: A selective review of two central issues, Rev. Modern Phys. 71 (1999) S346–S357.

[2] K. Wilson, Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture, Phys. Rev. B 9 (4) (1971) 3174. http://dx.doi.org/10.1103/PhysRevB.4.3174.

[3] K. Wilson, Renormalization group and critical phenomena. II. Phase-space cell analysis of critical behavior, Phys. Rev. B 9 (4) (1971) 3184. http://dx.doi.org/10.1103/PhysRevB.4.3184.

[4] K. Wilson, Problems in physics with many scales of length, Sci. Am. 241 (2) (1979) 158.

[5] R. Zwanzig, Memory effects in irreversible thermodynamics, Phys. Rev. 124 (06) (1961) 983.

[6] H. Mori, Transport, collective motion, and Brownian motion, Progr. Theoret. Phys. 33 (06) (1965) 423.

[7] K. Huang, Statistical Mechanics, Wiley, New York, 1987.

[8] H. Risken, The Fokker–Planck Equation: Methods of Solution and Applications, Springer-Verlag, New York, 1984.

[9] I. Snook, The Langevin and Generalised Langevin Approach to the Dynamics of Atomic, Polymeric and Colloidal Systems, Elsevier Science, 2007.

[10] L.D. Landau, E.M. Lifshitz, Fluid Mechanics: Course of Theoretical Physics Vol. 6, Butterworth-Heineman, Amsterdam, 2005.

[11] E. Nelson, Dynamical Theories of Brownian Motion, second ed., Princeton University Press, Princeton, 2006, Also available at: www.math.princeton.edu/~nelson/books.html.

[12] N. Vankampen, I. Oppenheim, Brownian motion as a problem of eliminating fast variables, Physica A 138 (1–2) (1986) 231–248. http://dx.doi.org/10.1016/0378-4371(86)90183-4. URL: http://linkinghub.elsevier.com/retrieve/pii/0378437186901834.

[13] H.A. Kramers, Brownian motion in a field of force and the diffusion model of chemical reactions, Physica 7 (1940) 284–304.

[14] J.E. Moyal, Stochastic processes and statistical physics, J. R. Stat. Soc. Lond. Ser. B 11 (1949) 150–210.

[15] D.O. Soares-pinto, W.A.M. Morgado, Brownian dynamics, time-averaging and colored noise, Physica A 365 (2006) 289–299. http://dx.doi.org/10.1016/j.physa.2006.01.063.

[16] D.O. Soares-Pinto, W.A.M. Morgado, Exact time-average distribution for a stationary non-Markovian massive Brownian particle coupled to two heat baths, Phys. Rev. E 77 (2008) 011103. http://dx.doi.org/10.1103/PhysRevE.77.011103.

[17] W.A.M. Morgado, D.O. Soares-pinto, Exact time-averaged thermal conductance for small systems: Comparison between direct calculation and Green-Kubo formalism, Phys. Rev. E 79 (2009) 051116. http://dx.doi.org/10.1103/PhysRevE.79.051116.

[18] W.A.M. Morgado, S.M.D. Queirós, Role of the nature of noise in the thermal conductance of mechanical systems, Phys. Rev. E 86 (2012) 041108. http://dx.doi.org/10.1103/PhysRevE.86.041108.

[19] W.A.M. Morgado, D.O. Soares-Pinto, Exact nonequilibrium work generating function for a small classical system, Phys. Rev. E 82 (2010) 021112. http://dx.doi.org/10.1103/PhysRevE.82.021112.

[20] W.A.M. Morgado, S.M. Duarte Queirós, D.O. Soares-Pinto, On exact time averages of a massive Poisson particle, J. Stat. Mech. Theory Exp. 2011 (06) (2011) P06010. http://dx.doi.org/10.1088/1742-5468/2011/06/P06010. URL: http://stacks.iop.org/1742-5468/2011/i=06/a=P06010?key=crossref.26ac05faadb506430fc976fab398c952.

[21] W.A.M. Morgado, T. Guerreiro, A study on the action of non-Gaussian noise on a Brownian particle, Physica A 391 (15) (2012) 3816–3827. http://dx.doi.org/10.1016/j.physa.2012.03.012. URL: http://linkinghub.elsevier.com/retrieve/pii/S0378437112002312.
[22] W.A.M. Morgado, S.M. Duarte Queirós, Thermostatistics of small nonlinear systems: Gaussian thermal bath Welles, Phys. Rev. E 90 (2014) 022110. http://dx.doi.org/10.1103/PhysRevE.90.022110.
[23] N.C. van Kampen, Stochastic Processes in Physics and Chemistry, North-Holland, Amsterdam, 1992.
[24] T.W. Duck, P.W. Marshall, E.J. Watson, First-passage times for the Uhlenbeck–Ornstein process, J. Phys. A: Math. Gen. 19 (1986) 3545–3558.
[25] R.F. Pawula, Phys. Rev. 162 (1967) 186.
[26] R.F. Pawula, IEEE Trans. Inform. Theory 13 (1967) 33.
[27] J. Marcinkiewicz, Math. Z. 44 (1939) 612.
[28] I. Oppenheim, K. Shuler, Phys. Rev. B 138 (1965) 1007. http://dx.doi.org/10.1103/PhysRevB.4.3184.
[29] P. Hanggi, H. Thomas, Z. Phys. B 26 (1977) 85.
[30] C. Gardiner, Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences, Springer-Verlag, Berlin, 1983.
[31] P.J. Smith, A recursive formulation of the old problem of obtaining moments from cumulants and vice versa, Amer. Statist. 49 (1995) 217–218.
[32] P. Hanggi, Correlation functions and masterequations of generalized (non-Markovian) Langevin equations*, Z. Phys. B 416 (1978) 407–416.
[33] P. Hanggi, P. Talkner, A remark on truncation schemes of cumulant hierarchies, J. Stat. Phys. 22 (1) (1980) 65–67. http://dx.doi.org/10.1007/BF01007988. URL: http://link.springer.com/10.1007/BF01007988.
[34] K. Kanazawa, T. Sano, T. Sagawa, H. Hayakawa, Minimal model of stochastic athermal systems: Origin of non-Gaussian noise, Phys. Rev. Lett. 114 (2015) 090601. http://dx.doi.org/10.1103/PhysRevLett.114.090601.