Nonadditive Entropy *Sq* and Nonextensive Statistical Mechanics: Applications in Geophysics and Elsewhere

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Abstract

The celebrated Boltzmann–Gibbs (BG) entropy, $S_{BG} = -k\sum_i p_i \ln p_i$, and associated statistical mechanics are essentially based on hypotheses such as ergodicity, *i.e.*, when ensemble averages coincide with time averages. This dynamical simplification occurs in classical systems (and quantum counterparts) whose microscopic evolution is governed by a ***positive*** largest Lyapunov exponent (LLE). Under such circumstances, relevant microscopic variables behave, from the probabilistic viewpoint, as (nearly) independent. Many phenomena exist, however, in natural, artificial and social systems (geophysics, astrophysics, biophysics, economics, and others) that violate ergodicity. To cover a (possibly) wide class of such systems, a generalization (nonextensive statistical mechanics) of the BG theory was proposed in 1988. This theory is based on nonadditive entropies such as $S_q = k\frac{1-\sum_i p_i^q}{q-1}$ $(S_1 = S_{BG})$. Here we comment some central aspects of this theory, and briefly review typical predictions, verifications and applications in geophysics and elsewhere, as illustrated through theoretical, experimental, observational, and computational results.

Key words: nonadditive entropy, nonextensive statistical mechanics, complex systems.

1. INTRODUCTION

Statistical mechanics constitutes, together with electromagnetism (basically Maxwell equations) and classical (basically Newton's law), quantum (basically Schroedinger equation), and relativistic (basically Einstein's special and general relativity) mechanics, one of the pillars of contemporary physics. As its name indicates, ***statistical mechanics*** is essentially constructed from two ingredients, namely ***mechanics*** (including the electro-magnetic forces) and ***theory of probabilities*** (see Fig. 1). From this level, which is to be considered as from first principles, we may construct concepts such as energy and entropy, including their operational mathematical expressions. How to find the expressions for the energy is described in all good textbooks of mechanics (classical, quantum, relativistic). What concerns a similar task for the entropy, it has proved along 140 years (since the 1860-1870's, when Clausius and Boltzmann first focused on this issue) to be a particularly delicate matter. In principle, it goes like this. Guided by thermodynamics, we
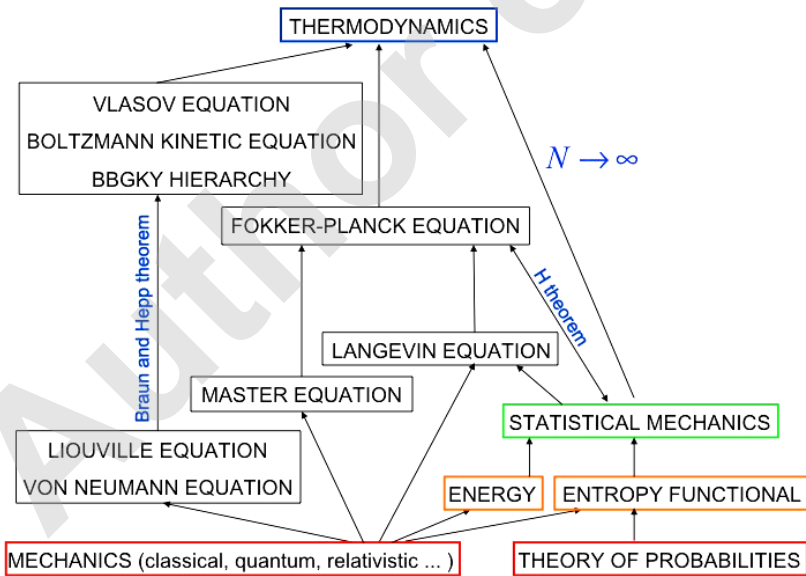

Fig. 1. Schematic and non-exhaustive connections that exist related with statistical mechanics. In red we have what we may call the first-principles or ***microscopic*** level, from where we can, in one or another, derive the concepts of energy and entropy (in orange). These concepts lead to statistical mechanics (in green), which, in turn, connects to thermodynamics or ***macroscopic*** level (in blue) for large systems. Many ***mesoscopic*** levels exist as well, of which we have indicated here only the most common ones. Colour version of this figure is available in electronic edition only.

would like the entropy S of a macroscopic system to be ***extensive***, *i.e.*, $S(N) \propto N(N \to \infty)$, N being the number of elements of the system. We expect this entropic extensivity to be valid for both short-and long-range interactions. For short-range interactions, it is well known that it should be so, as explained in any good textbook of thermodynamics. The question is more subtle for long-range interactions, for which the total energy U becomes nonextensive (more precisely, $U(N)$ is expected to increase faster than N for large N). But even in this case, the thermodynamic entropy should remain extensive, as lengthily argued and verified in the literature (see *e.g.*, Section 3.3.1 of Tsallis 2009a). Accepting this general thermodynamic demand, the next relevant question is what mathematical connection between S and its probabilistic expression (in terms of the admissible microscopic configurations) adequately takes into account the correlations existing between the N elements of the system in such a way that $S(N)$ is extensive. Let us illustrate this crucial point in what follows.

Assume that we have a system whose total number $W(N)$ of admissible microscopic configurations are equally probable, and satisfies

$$W(N) \propto \mu^N \ (N \to \infty, \quad \mu > 1),\tag{1}$$

Such a hypothesis corresponds to probabilistic independence (or quasi-independence) of the N elements of the system, since $W(N+1) \sim \mu W(N)$.

We know of course that the Boltzmann-Gibbs (BG) entropy is given (for discrete variables) by

$$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i \quad \left(\sum_{i=1}^{W} p_i = 1 \right),\tag{2}$$

hence, for equal probabilities, we have

$$S_{BG} = k \ln W .\tag{3}$$

If we introduce Eq. (1) in Eq. (3), we obtain

$$S_{BG}(N) = k \ln W(N) \propto N .\tag{4}$$

In other words, the BG formula precisely yields the desired extensivity for the entropy.

Let us assume now a system whose elements are strongly correlated in such a way that the number of equally probable admissible configurations (*i.e.*, configurations whose probability is nonzero) satisfies

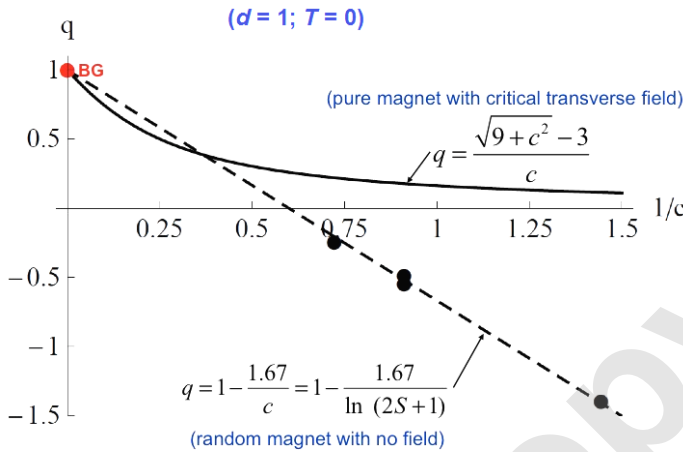$$W(N) \propto N^{\rho} \quad (N \to \infty, \quad \rho > 0) .\tag{5}$$

Fig. 2. Dependence of q_{ent} on the central charge c of pure (Caruso and Tsallis 2008) and random (Saguia and Sarandy 2010) one-dimensional magnets undergoing quantum phase transitions at zero temperature, where the entire strongly entangled N-system is in its ground state (hence corresponding to a vanishing entropy since the ground state is a pure state), in contrast to the L-subsystem which is in a mixed state (hence corresponding to a nonvanishing entropy). For this value of q, the block nonadditive entropy S_q is extensive, whereas its additive BG entropy is nonextensive. Notice that, for the pure magnet, we have $q_{\text{ent}} \in [0, 1]$, whereas, for the random magnet, we have $q_{\text{ent}} \in (-\infty, 1]$. Both cases recover, in the $c \to \infty$ limit, the BG value $q_{\text{ent}} = 1$. These examples definitively clarify that additivity and extensivity are different properties. The only reason for which they have been confused (and still are confused in the mind of not few scientists!) is the fact that, during 140 years, the systems that have been addressed are simple, and not complex, thermodynamically speaking. For such non-pathological systems, the additive BG entropy happens to be extensive, and is naturally the one that should be used. Colour version of this figure is available in electronic edition only.

We cannot use S_{BG} in this case, since it implies $S_{BG}(N) \propto \ln N$, which violates thermodynamics. But we may consider instead the following generalized expression (Tsallis 1988):

$$S_q = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} = -k \sum_{i=1}^{W} p_i^q \ln_q p_i = k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i} \quad \left(\sum_{i=1}^{W} p_i = 1 \right), \tag{6}$$

where $\ln_q z \equiv \frac{z^{1-q} - 1}{1 - q}$ $(\ln_1 z = \ln z)$. We straightforwardly verify the $S_1 = S_{BG}$, and also that, for equal probabilities,

$$S_q = k \ln_q W . \tag{7}$$

If we introduce, within this entropy, expression (5), we obtain

$$S_q(N) = k \frac{[W(N)]^{1-q} - 1}{1-q} \propto N^{\rho(1-q)} \,. \tag{8}$$

Consequently, if we choose

$$q = 1 - \frac{1}{\rho} \,, \tag{9}$$

we obtain $S_{1-\frac{1}{\rho}}(N) \propto N$, in agreement with thermodynamics. This is the basic reason why the BG entropy must be adequately replaced in cases where strong correlations exist in the system. It is appropriate to mention here that all kinds of asymptotic mathematical behaviors can in principle exist for $W(N)$. For those, other entropic functionals become necessary in order to have extensivity. In the present work, however, we focus on Eq. (5), which represents in fact a quite wide class of natural, artificial and social systems, as we shall see later on. Verifications on other probabilistic and physical models do exist in the literature which illustrates the fact that S_q is extensive for special values of q, which characterize the universality class of the system (see Tsallis 2004, Tsallis *et al.* 2005, and Caruso and Tsallis 2008). In Fig. 2 we exhibit that special value of q for fully quantum-entangled pure and random magnetic systems.

Let us close this section by emphasizing that ***nonextensivity*** must be well distinguished from ***nonadditivity*** (Penrose 1970) Indeed, an entropy is said additive if, for two probabilistically independent systems *A* and *B*, we verify that $S(A + B) = S(A) + S(B)$. We can easily establish that

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\frac{S_q(A)}{k}\frac{S_q(B)}{k} \,. \tag{10}$$

Therefore, S_{BG} is additive, whereas, for $q \neq 1$, S_q is nonadditive.

2. EXTREMIZATION OF S_q AND q-GENERALIZED CENTRAL LIMIT THEOREMS

Let us now focus on the case where the random variable **x** is a continuous dimensionless *D*-dimensional one. The *q*-entropy is then given by[1]

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[1]We should naturally have in mind that this type of expression cannot be used in thermo-statistics for extremely low temperatures, where the quantum nature of natural systems must be taken into account. In other words, if $p(\mathbf{x})$ is too thin, *i.e.*, too close to a Dirac delta $\delta(\mathbf{x} - \mathbf{x}_0)$, Eq. (11) will become negative ($\forall q$), which is inadmissible for entropy.

$$S_q = k \frac{1 - \int d\mathbf{x} \left[p(\mathbf{x}) \right]^q}{q-1} = -k \int d\mathbf{x} \left[p(\mathbf{x}) \right]^q \ln_q p(\mathbf{x})$$

$$= k \int d\mathbf{x} \, p(\mathbf{x}) \ln_q \frac{1}{p(\mathbf{x})} \quad \left(\int d\mathbf{x} \, p(\mathbf{x}) = 1 \right). \tag{11}$$

We further assume that we have a ***cost function*** $E(\mathbf{x})$ (*e.g.*, the Hamiltonian of the total system in mechanical systems), and that we know the q-mean value (which characterizes the **width** of the distribution $p(\mathbf{x})$ even when its standard mean value $\langle E(\mathbf{x}) \rangle_1$ diverges (Budde *et al.* 2001, Tsallis *et al.* 2009)) of this cost function, *i.e.*,

$$\langle E(\mathbf{x}) \rangle_q \equiv \frac{\int d\mathbf{x} \, E(\mathbf{x}) \left[p(\mathbf{x}) \right]^q}{\int d\mathbf{x} \left[p(\mathbf{x}) \right]^q}. \tag{12}$$

Notice that this quantity is finite up to the same value of q for which the norm $\int d\mathbf{x} \, p(\mathbf{x})$ itself is finite.

If our system is a dynamical one with a physically relevant stationary state (a frequent case), this state is the one which, under the constraint (12), extremizes S_q. It is quite straightforward to verify that the maximizing distribution is as that given by Tsallis (1988), Curado and Tsallis (1991), and Tsallis *et al.* (1998)

$$p_q(\mathbf{x}) = \frac{e_q^{-\beta E(\mathbf{x})}}{\int dx \, e_q^{-\beta E(\mathbf{x})}} \quad (\beta > 0), \tag{13}$$

where $e_q^z (e_1^z = e^z)$ is the inverse function of $\ln_q z$, *i.e.*, $e_q^z \equiv [1 + (1-q)z]_+^{\frac{1}{1-q}}$, with $[z]_+ = z$ if $z \geq 0$, and zero otherwise.

If $\mathbf{x}$ is a $D = 1$ continuous variable x, and $E(x) \propto x$, the constraint (12) becomes the value of $\langle x \rangle_q$, hence

$$p_q(x) = \frac{e_q^{-\beta_1 x}}{\int dx \, e_q^{-\beta_1 x}} \quad (\beta_1 > 0), \tag{14}$$

This distribution is normalizable for $q < 2$, and has a finite mean value $\langle x \rangle_1$ for $q < 3/2$. For $q \geq 1$ it has an infinite support, whereas it is finite for $q < 1$.

If $\langle x \rangle_q = 0$, and we happen to know $\langle x^2 \rangle_q$, this quantity becomes the constraint (12), and can be used to characterize the width. The extremization of S_q then yields

$$p_q(x) = \frac{e_q^{-\beta_2 x^2}}{\int dx \, e_q^{-\beta_2 x^2}} \quad (\beta_2 > 0), \tag{15}$$

which from now on will be referred to as q-Gaussian distribution[2]. It is normalizable for $q < 3$, and has a finite variance $\langle x^2 \rangle_1$ for $q < 5/3$[3].

For $q \geq 1$ it has an infinite support, whereas it is finite for $q < 1$. For $q > 1$, q-Gaussians asymptotically decay as power-laws (more precisely like $x^{-2/(q-1)}$). However, they are quite different from the Lévy distributions, which also decay like power-laws (the only case in which q-Gaussians and Lévy distributions coincide is for $q = 2$, which corresponds to the Cauchy–Lorentz distribution). Quite frequently, in the literature, any distribution decaying like a power-law is referred to as a "Lévy distribution", which constitutes a rather regrettable mistake.

Let us mention at this stage an interesting mesoscopic property of q-exponentials and q-Gaussians. We consider the following nonlinear diffusion-relaxation equation:

$$\frac{\partial p(x, t)}{\partial t} = D \frac{\partial^2 \left[p(x, t) \right]^{2-q}}{\partial x^2} - R \left[p(x, t) \right]^q \left(\int \mathrm{d}x \, p(x, 0) = 1 \right), \tag{16}$$

where D and R are constant phenomenological coefficients. If $D = 0$ we have

$$p(x, t) = p(x, 0) \, e_q^{-Rt} (\forall x) \,, \tag{17}$$

which has the form of Eq. (14) with $\beta_1 = R$. If we have instead $R = 0$ we obtain (Plastino and Plastino 1995, Tsallis and Bukman 1996) the form of Eq. (15)

$$p(x, t) \propto e_q^{-\beta_2(t)x^2} \,, \tag{18}$$

where β_2 is related with D. In other words, q-exponentials and q-Gaussians are exact solutions of basic nonlinear diffusion-relaxation equations. They can be shown to also provide exact stationary states of similar though inhomogenous equations (dos Santos and Tsallis 2010).

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[2]Since long known in plasma physics under the name ***suprathermal*** or κ ***distributions*** (Rios and Galvão 2011) if $q > 1$, and equal to the Student's t-distributions (de Souza and Tsallis 1997) for special rational values of $q > 1$; for special rational values of $q < 1$, it coincides with the so-called r-distributions (de Souza and Tsallis 1997). They are also occasionally referred to as generalized Lorentzians (Treumann 1998).

[3]If x is a D-dimensional real vector, normalizability mandates that $\int_0^\infty \mathrm{d}x \, x^{D-1} e_q^{-\beta x^2}$ converges, hence $q < \frac{D+2}{D}$. If, in addition to that, the system has a density of states $\phi(x)$ which diverges like x^δ for $x \to \infty$ (a quite frequent case), then normalizability mandates $\int_0^\infty \mathrm{d}x \, x^{D-1} \phi(x) e_q^{-\beta x^2}$ converges, hence $q < \frac{D+\delta+2}{D+\delta}$. Similarly, the finiteness of the second moment mandates that $\int_0^\infty \mathrm{d}x \, x^{D+1} \phi(x) e_q^{-\beta x}$ converges, hence $q < \frac{D+\delta+4}{D+\delta+2}$.

We have briefly reminded above that these distributions appear simultaneously as those which optimize (maximize for $q > 0$, and minimize for $q < 0$), under appropriate constraints, the nonadditive entropy S_q, and as those which exactly solve nonlinear/inhomogeneous diffusion-relaxation equations. Let us conclude by mentioning another remarkable property, namely that q-Gaussians constitute attractors in the sense of the classical central limit theorem (CLT). This theorem basically states that, if we consider the sum $S_N = \sum_{i=1}^{N} X_i$ of N **independent** (or nearly independent in some sense) random variables $\{X_i\}$, each of them having a **finite** variance, this sum converges for $N \to \infty$, after appropriate centering and rescaling, to a Gaussian. This most important theorem can be proved in a variety of manners and under a slightly different hypothesis. One of those standard proofs uses the Fourier transform, which has been q-generalized (Umarov *et al.* 2008, 2010).

Around 2000 (Bologna *et al.* 2000), q-Gaussians have been conjectured (see details in Tsallis 2005) to be attractors in the CLT sense whenever the N random variables that are being summed are strongly correlated in a specific manner. The conjecture was recently proved in the presence of q-independent variables (Umarov *et al.* 2008, 2010). The proof presented in Umarov *et al.* (2008) is based on a q-generalization of the Fourier transform, denoted as q-Fourier transform, and the theorem is currently referred to as the q-CLT. The validity of this proof has been recently challenged by Hilhorst (2009, 2010). His criticism is constructed on the inexistence of inverse q-Fourier transform for $q > 1$, which he illustrates with counterexamples. The inverse, as used in the Umarov *et al.* (2008) paper, indeed does not exist in general, which essentially makes the proof presented in Umarov *et al.* (2008) a proof of existence, but not of uniqueness. The q-generalization of the inverse Fourier transform appears then to be a quite subtle mathematical problem if $q \neq 1$. It has nevertheless been solved recently (Jauregui and Tsallis 2011) and further considerations can be seen in Jauregui *et al.* (2010, 2011), related to q-moments (Budde *et al.* 2001, Tsallis *et al.* 2009, Rodríguez and Tsallis 2010). Work is in progress attempting to transform the existence proof in Umarov *et al.* (2008, 2010) into a uniqueness one. It is fair to say that, at the present moment, a gap exists in the complete proof (not necessarily in the thesis) of the q-CLT as it stands in Umarov *et al.* (2008). In the meanwhile, several other forms (Vignat and Plastino 2007, 2009, Hahn *et al.* 2010) of closely related q-generalized CLT's have already been published which do not use the inverse q-Fourier transform (see Fig. 3).

Probabilistic models have been formulated (Rodríguez *et al.* 2008, Hanel *et al.* 2009) which, in the $N \to \infty$ limit, yield q-Gaussians. These models are scale-invariant, which might suggest that q-independence implies (either strict or asymptotic) scale-invariance, but this is an open problem at the

	$q = 1$ [*independent*]	$q \neq 1$ (*i.e.*, $Q \equiv 2q - 1 \neq 1$) [*globally correlated*]
$\sigma_Q < \infty$ ($\alpha = 2$)	$F(x) = Gaussian\ G(x)$ *with same* σ_1 of $f(x)$ Classic CLT	$F(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x)$, *with same* σ_Q *of* $f(x)$ $G_q(x) \sim \begin{cases} G(x) \\ \quad if\ \|x\| \ll x_c(q, 2) \\ f(x) \sim C_q/\|x\|^{2/(q-1)} \\ \quad if\ \|x\| \gg x_c(q, 2) \end{cases}$ *with* $\lim_{q \to 1} x_c(q, 2) = \infty$ S. Umarov *et al.*, Milan J. Math. 76, 307 (2008)
$\sigma_Q \to \infty$ ($0 < \alpha < 2$)	$F(x) = Lévy\ distribution\ L_\alpha(x)$ *with same* $\|x\| \to \infty$ *behavior* $L_\alpha(x) \sim \begin{cases} G(x) \\ \quad if\ \|x\| \ll x_c(1, \alpha) \\ f(x) \sim C_\alpha/\|x\|^{1+\alpha} \\ \quad if\ \|x\| \gg x_c(1, \alpha) \end{cases}$ *with* $\lim_{\alpha \to 2} x_c(1, \alpha) = \infty$ Lévy-Gnedenko CLT	$F(x) = L_{q,\alpha}$, *with same* $\|x\| \to \infty$ *asymptotic behavior* $L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)}, \alpha}(x) \sim C^*_{q,\alpha}/\|x\|^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \quad (\textit{intermediate regime}) \\ G_{\frac{2\alpha q - \alpha + 3}{\alpha+1}, 2}(x) \sim C^L_{q,\alpha}/\|x\|^{(1+\alpha)/(1+\alpha q - \alpha)} \\ \quad (\textit{distant regime}) \end{cases}$ S. Umarov *et al.*, J. Math. Phys. 51, 033502 (2010)

Fig. 3. Central limit theorems (CLT) for $q \geq 1$: $N^{1/[\alpha(2-q)]}$-scaled attractor $F(x)$ when summing $N \to \infty$ identical q-independent random variables with symmetric distribution $f(x)$ with $\sigma_Q \equiv \int dx\, x^2[f(x)]^Q / \int dx\,[f(x)]^Q$ ($Q \equiv 2q - 1$). All the attractors of these theorems asymptotically decay as power-laws, except of the classical CLT. See details in Umarov *et al.* (2008, 2010). Colour version of this figure is available in electronic edition only.

present time. However, definitively, scale-invariance does not imply q-independence. Indeed, (strictly or asymptotically) scale-invariant probabilistic models are known (Moyano *et al.* 2006, Thistleton *et al.* 2009) which do not yield q-Gaussians, but other limiting distributions instead, some of which have been proved to be amazingly close to q-Gaussians (Hilhorst and Schehr 2007).

Some of the predictions, verifications and applications of this q-generalized theory are briefly reviewed in the rest of the present paper, which is based in fact on various previous books and reviews (Tsallis

2009a, b, 2011, Gell-Mann and Tsallis 2004, Boon and Tsallis 2005), parts of which are here followed/reproduced for simplicity and self-completeness.

3. APPLICATIONS

In this section we briefly and non exhaustively review various predictions, verifications and applications of q-exponentials and q-Gaussians through analytical, experimental, observational and computational methods in natural, artificial and social systems that are available in the literature (see at http://tsallis.cat.cbpf.br/biblio.htm for full bibliography). The present list enriches the one recently presented in Tsallis (2011). Several of these applications concern, as it can be checked in what follows, geophysical phenomena, especially earthquakes and similar ones.

❑   The velocity distribution of (cells of) **Hydra viridissima** follows a q-Gaussian probability distribution function (PDF) with $q \simeq 3/2$ (Upadhyaya *et al*. 2001). Anomalous diffusion has been independently measured as well (Upadhyaya *et al*. 2001), and an exponent $\gamma \simeq 4/3$ has been observed (where the squared space x^2 scales with time t like t^γ). Therefore, within the error bars, the prediction $\gamma = 2/(3-q)$ (Tsallis and Bukman 1996) is verified in this system.

❑   The velocity distribution of the point defects in defect turbulence, as well as its corresponding anomalous diffusion, have been measured (Daniels *et al*. 2004). The results suggest a q-Gaussian PDF with $q \simeq 3/2$, and $\gamma \simeq 4/3$ which constitutes another verification of the prediction $\gamma = 2/(3-q)$ (Tsallis and Bukman 1996).

❑   The velocity distribution of cold atoms in a dissipative optical lattice was predicted (Lutz 2003) to be a q-Gaussian with $q = 1 + 44\frac{E_r}{U_0}$, where E_r and U_0 are parameters related to the optical lattice potential. This prediction was verified three years later, both in the laboratory and with quantum Monte Carlo techniques (Douglas *et al*. 2006, Baris Bagci and Tirnakli 2009).

❑   Computational simulations of the velocity distribution, and of the associated anomalous diffusion, during silo drainage suggest $q \simeq 3/2$ and $\gamma \simeq 4/3$ (Arevalo *et al*. 2007), once again satisfying the prediction $\gamma = 2/(3-q)$ (Tsallis and Bukman 1996).

❑   The velocity distribution in a driven-dissipative 2D dusty plasma was found to be of the q-Gaussian form, with $q = 1.08 \pm 0.01$ and $q = 1.05 \pm 0.01$ at temperatures of 30000 and 61000 K, respectively (Liu and Goree 2008).

❑ The spatial (Monte Carlo) distributions of a trapped $^{136}Ba^+$ ion cooled by various classical buffer gases at 300 K was verified to be of the q-Gaussian form, with q increasing from close to unity to about 1.9 when the mass of the molecules of the buffer increases from that of He to about 200 (De Voe 2009).

❑ The distributions of price returns and stock volumes at the New York and NASDAQ stock exchanges are well fitted by q-Gaussians and q-exponentials, respectively (Borland 2002, Duarte Queirós 2005a). The volatilities predicted within this approach fit well the real data. Various other economical and financial applications are available (Borges 2004, Mattedi *et al.* 2004, Borland and Bouchaud 2004, Borland 2005, Duarte Queirós 2005b, Takahashi *et al.* 2007, Ferrero 2011), including those associated with extreme values and risk (Ludescher *et al.* 2011).

❑ The Bak-Sneppen model of biological evolution exhibits a time-dependence of the spread of damage which is well approached by a q-exponential with $q < 1$ (Tamarit *et al.* 1998).

❑ The distributions of returns in the Ehrenfest's dog-flea model exhibit a q-Gaussian form (Bakar and Tirnakli 2010).

❑ The distributions of returns in the coherent noise model are well fitted with q-Gaussians where q is analytically obtained through $q = (2 + \tau)/\tau$, τ being the exponent associated with the distribution of sizes of the events (Celikoglu *et al.* 2010).

❑ The distributions of returns of the avalanche sizes in the self-organized critical Olami–Feder–Christensen model, as well as in real earth-quakes exhibit a q-Gaussian form (Caruso *et al.* 2007).

❑ The distributions of angles in the HMF model approaches as time evolves towards a q-Gaussian form with $q \simeq 1/5$ (Moyano and Anteneodo 2006).

❑ Experimental measurements of the turbulence in pure electron plasma are analytically reproduced with $q = 1/2$ (Boghosian 1996).

❑ The relaxation in various paradigmatic spin-glass substances through neutron spin echo experiments is well reproduced by q-exponential forms with $q > 1$ (Pickup *et al.* 2009).

❑ The fluctuating time dependence of the width of the ozone layer over Buenos Aires (and, presumably, around the Earth) yields a q-triplet with $q_{sen} < 1 < q_{stat\ state} < q_{rel}$ (Ferri *et al.* 2010).

❑ Diverse properties for conservative and dissipative nonlinear dynamical systems are well described within q-statistics (Anteneodo and Tsallis 1998, Lyra and Tsallis 1998, Borges *et al.* 2002, Añaños and Tsallis 2004, Baldovin and Robledo 2004, Añaños *et al.* 2005, Tirnakli *et al.* 2007,

2009, Pluchino *et al*. 2008, Afsar and Tirnakli 2010, Leo *et al*. 2010, Fuentes *et al*. 2011, Frank 2011).

❑ The degree distribution of (asymptotically) scale-free networks is numerically calculated and is well approached by a *q*-exponential distribution (Thurner and Tsallis 2005).

❑ The tissue radiation response follows a *q*-exponential form (Sotolongo-Grau *et al*. 2010).

❑ The overdamped motion of interacting particles in type II superconductors is analytically shown to follow, at vanishing temperature, a *q*-Gaussian with $q = 0$. Moreover, the entropy is the nonadditive one associated with this value of *q* (Andrade *et al*. 2010).

❑ Experimental and simulated molecular spectra due to the rotational population in plasmas are frequently interpreted as two Boltzmann distributions corresponding to two different temperatures. These fittings involve three fitting parameters, namely the two temperatures and the relative proportion of each of the Boltzmann weights. It has been shown (Reis *et al*. 2011) that equally good fittings can be obtained with a single q-exponential weight, which has only two fitting parameters, namely *q* and a single temperature.

❑ High energy physics has been since more than one decade handled with *q*-statistics (Bediaga *et al*. 2000). During the last decade various phenomena, such as the flux of cosmic rays and others, have been shown to exhibit relevant nonextensive aspects (Wilk and Wlodarczyk 2009, Biró *et al*. 2009). The distributions of transverse momenta of hadronic jets outcoming from proton-proton collisions (as well as others) have been shown to exhibit *q*-exponentials with $q \simeq 1/1$. These results have been obtained at the LHC detectors CMS, ATLAS and ALICE (CMS Collaboration 2010, ALICE Collaboration *et al*. 2011, ATLAS Collaboration 2010), as well as at SPS and RHIC in Brookhaven (Shao *et al*. 2010). Predictions for the rapidities in such experiments have been advanced as well. These results stimulate an interesting possible dialog between nonextensive statistics and quantum chromodynamics (QCD).

❑ Various astrophysical systems exhibit nonextensive effects (Carvalho *et al*. 2008, Esquivel and Lazarian 2010, Livadiotis and McComas 2009, Livadiotis *et al*. 2011).

❑ Analysis of the magnetic field in the solar wind plasma using data from Voyager 1 and Voyager 2 strongly suggests nonextensive effects (Burlaga and Viñas 2005, Burlaga and Ness 2011).

❑ Various geophysical applications exhibit nonextensive effects (Tirnakli and Abe 2004, Abe *et al.* 2005, Papadimitriou *et al.* 2008, Vallianatos 2009, Balasis *et al.* 2009, Telesca 2010a, b; Vallianatos and

Sammonds 2010, Eftaxias 2010, Vallianatos 2011, Vallianatos *et al.* 2011a, b).

❏   Nonlinear generalizations of the Schroedinger, the Klein-Gordon and the Dirac equations have been impl emented which admit *q*-plane wave solutions as free particles, *i.e.*, solutions of the type $e_q^{i(kx-\omega t)}$ (Nobre *et al.* 2011), with the energy given by $E = \hbar\omega$ and the momentum given by $\vec{p} = \hbar\vec{k}$, $\forall q$. The nonlinear Schroedinger equation yields $E = p^2/2m$ ($\forall q$), and the nonlinear Klein-Gordon and Dirac equations yield the Einstein relation $E^2 = m^2c^4 + p^2c^2$ ($\forall q$).

❏   Phenomena in linguistics such as Zipf law and the frequency of words in various languages and literary styles (Montemurro 2004).

❏   Statistics of citations of scientific and technological papers (Tsallis and Portes de Albuquerque 2000, Anastasiadis *et al.* 2010).

❏   Processing of medical signals such as those emerging in epileptic crisis (Tong *et al.* 2002, Rosso *et al.* 2003).

❏   Processing of medical and other images (Portes de Albuquerque *et al.* 2004, Shi *et al.* 2009, Mohanalin *et al.* 2010).

❏   Global optimization algorithms generalizing simulated annealing and others (Tsallis and Stariolo 1996, Serra *et al.* 1997, Anastasiadis and Magoulas 2004).

❏   Diversified applications in theoretical chemistry (Mundim and Tsallis 1996, Andricioaei and Straub 1996, Hansmann *et al.* 1997, Gutterres *et al.* 1999, Straub and Andricioaei 1999, Fuks *et al.* 2001, Okamoto and Hansmann 2001, Moret *et al.* 2006, Kim and Straub 2009).

❏   Cognitive psychology in relation with learning and remembering (Tsallis *et al.* 2003, Takahashi *et al.* 2009, Wedemann *et al.* 2009).

❏   Astronomical systems (Carvalho *et al.* 2009).

The systematic study of metastable or long-living states in long-range versions of magnetic models such as the finite-spin Ising (Nobre and Tsallis 1995) and Heisenberg (Caride *et al.* 1983) ones, or in hydrogen-like atoms (Lucena *et al.* 1995, Oliveira-Neto *et al.* 2007), might provide further hints and applications.

4.   FINAL REMARKS

The Boltzmann-Gibbs entropy and exponential weight have been generalized, during the last two decades, in various manners (Beck and Cohen 2003, Hanel and Thurner 2011, Tempesta 2011, Hanel *et al.* 2011; see further details in Tsallis 2009a). These various manners follow essentially from the 1988 proposal that we have focused on in this brief review. The correspond-

ing entropy is noted S_q and it is nonadditive; for a wide class of systems, it should be extensive for a special value of the index q, which reflects the nature of strong correlations that the elements of the system have. The corresponding thermostatistics is currently referred to as nonextensive statistical mechanics (the word ***nonextensive*** stands here to reflect the fact that those systems typically have an internal energy which grows faster than the number of elements N).

The applicability of these concepts has been illustrated in the previous section through analytical, numerical, experimental and observational results. Nevertheless, very many interesting questions still remain as opened issues. For example: Under what conditions q-independence and scale-invariance co-exist? Under what conditions the present generalized thermostatistics is compatible with classical thermodynamics? How well can be described with a q-exponential density matrix the mixed state of the block whose entropy has been discussed in Caruso and Tsallis (2008)? Further analysis of these and other points would be very welcome. Finally, since the present special issue is also dedicated to natural hazards, let us suggest that the (all important) evaluation of their risks could perhaps benefit from analysis done along the lines that have proved useful for financial systems, as illustrated in Ludescher *et al*. (2011).

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