



Critical exponents for Fréedericksz transition in nematics between concentric cylinders

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ARTICLE INFO

Article history:

Received 26 August 2009

Available online 13 November 2009

PACS:

61.30.Gd

64.70.Md

64.60.Fr

Keywords:

Liquid crystals

Fréedericksz transition

Critical exponents

ABSTRACT

The equilibrium tilt angle profile in a cell limited by two concentric cylinders filled with nematic liquid crystals is determined for strong homeotropic anchoring at the surfaces. The anchoring condition is such that the nematic director is perpendicular to the cylinder axes and a radial nonuniform electric field is applied to investigate a Fréedericksz transition. The distortions induced by the field remain in the plane perpendicular to the cylinder axes, and a threshold field is analytically determined indicating a transition from a pure splay to a splay-bend conformation of the director. It is shown that this transition can be induced by the thickness of the region between the two cylinders, and can be detected even in the absence of an external field. If the maximum value of the tilt angle is assumed as an order parameter, its behavior near to the transition can be used to obtain the critical exponent, which is the same as the one obtained in the mean field approximation. These results are indications that nontrivial consequences may occur when complex fluids are subject to non-planar geometries.

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1. Introduction

The liquid crystal alignment between two concentric cylinders has been analyzed by different authors since its original proposition by Meyer [1–7]. Our aim is to consider the problem in a different perspective: first we solve it in a geometry that is different from the one considered in the previous work and then we focus our attention on the order parameter critical exponent. Specifically, we analyze the molecular orientation of a nematic sample contained between two cylinders of radius a and b , such that $b = \rho a > a$, whose surfaces are homogeneous. In Ref. [3], the Fréedericksz transition was investigated for a nonuniform field, for homeotropic anchoring, by searching an “escaped” splay-bend configuration as shown in Fig. 1(a). In this work, we investigate the existence of a critical field for the distortion as shown in Fig. 1(b) and investigate analytically the critical phenomenon. The initial alignment is purely radial, i.e., a conformation of the director profile exists that corresponds to a pure splay deformation. In the presence of the electric field, for nematic liquid crystals with a negative dielectric anisotropy ($\epsilon_a = \epsilon_{\parallel} - \epsilon_{\perp} < 0$), where \parallel and \perp refer to the direction of \vec{n} , a bend distortion can be found in the system. This geometry was considered by de Gennes [2], in the one-constant approximation, and by Williams [4] with unequal elastic constants in the absence of the electrical field, with the deformations caused by a pretilt. Similar problems have been considered in connection with Fréedericksz transition in confined geometries [8–11]. The problem we face here is therefore a generalization of the model treated in Refs. [2,4], but now taking into account the effect of an external field and investigating the exact problem near to the transition.

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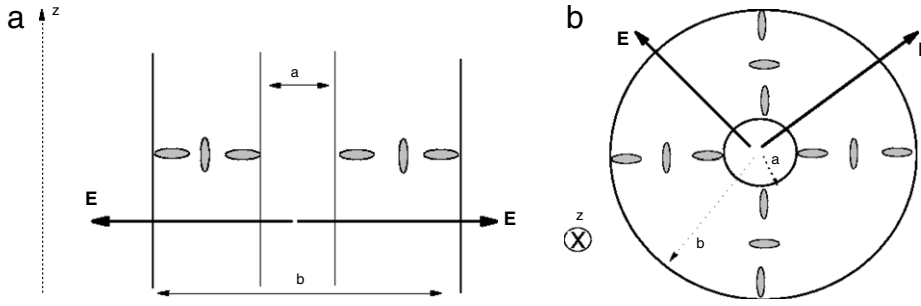


Fig. 1. Nematic sample in a geometry formed by two concentric cylinders. (a) The director “escapes” to the z -direction, perpendicular to a nonuniform radial electric field. (b) The director lies in the polar plane, i.e., the plane perpendicular to the cylinder axes along z .

2. Fréedericksz transition

The director is defined as $\vec{n} = \cos \psi \hat{r} + \sin \psi \hat{\theta}$, where ψ is the angle between \vec{n} and \hat{r} in the polar plane. For the geometry shown in Fig. 1(b), the problem has to be solved with the boundary conditions $\psi(a) = \psi(b) = 0$, considering strong anchoring, i.e., the molecules are fixed at the surface. We assume that the electric field is the one generated by a difference of potential $V = V(a) - V(b)$ between the cylinders, as in Ref. [3], i.e., $\vec{E} = \frac{V}{(r \ln \rho)} \hat{r}$. By neglecting the flexoelectric contribution, the total elastic energy density is composed of the Frank expression (in which the twist term as well as the saddle-splay term are absent) with a contribution coming from the dielectric coupling. In this manner, the total free energy per unit length of the cylinder, in the limit of small distortion ($\psi \ll 1$), and neglecting terms independent of ψ , is given by

$$F = \pi K_{33} \int_a^b \left[r \left(\frac{d\psi}{dr} \right)^2 - \frac{h}{r^2} \psi^2 \right] dr, \quad (1)$$

in which we have introduced the dimensionless quantities

$$h = k - \frac{\epsilon_a V^2}{K_{33} \ln^2 \rho} \quad \text{and} \quad k = \frac{K_{11}}{K_{33}} - 1,$$

with K_{11} and K_{33} , being, respectively, the elastic constants of splay and bend.

The transformation $r = ae^x$ (that will be used hereafter) permits us to rewrite (1) in the simple form:

$$F = \pi K_{33} \int_0^{\ln \rho} \left[\left(\frac{d\psi}{dx} \right)^2 - h \psi^2 \right] dx. \quad (2)$$

By minimizing F , as given above, we obtain

$$\frac{d^2 \psi}{dx^2} + h \psi = 0, \quad (3)$$

whose solution satisfying the boundary conditions is

$$\psi(x) = \psi_M \sin \left[\frac{l\pi}{\ln \rho} x \right] \quad (4)$$

if $(l\pi)^2 / \ln^2 \rho = h$, for $l = 1, 2, 3, \dots$. The quantity ψ_M is the maximum amplitude of $\psi(x)$, and, in the approximation we are working, can be considered as very small; it corresponds to the value of ψ in the middle points between the cylindrical surfaces when $l = 1$, i. e., $\psi[(\ln \rho)/2] = \psi_M$.

The situation of the lower energy corresponds to $l = 1$. By using the definition of h (with $h \geq 0$), we obtain for the critical voltage in the cylinder geometry the general expression

$$V_c^2 = \frac{\pi^2 K_{33}}{|\epsilon_a|} \left(1 - \frac{\pi^2 k}{\ln^2 \rho} \right). \quad (5)$$

This is the critical voltage for (cylindrical) Fréedericksz transition. If a voltage $V < V_c$ is applied, the elastic energy does not permit any deviation of non-distorted configuration; if, on the other hand, $V > V_c$, we can obtain a distorted situation, and the director profile is given by Eq. (4), with $l = 1$.

An important feature of the sample in this geometry is the existence of a Fréedericksz transition even in the absence of the electric field. This phenomenon was theoretically predicted by Williams and Halperin for a sample in a different geometry [3]; there, this phenomenon is defined as *Fréedericksz-like transition*. In fact, if $V_c = 0$, the relation (5) gives a threshold thickness:

$$\ln \rho_c = \frac{\pi}{\sqrt{k}}, \tag{6}$$

which suggests the existence of a distorted situation without the application of an electric field. Nevertheless, it is necessary that $k > 0$, which implies $K_{11} > K_{33}$, a situation not common for usual liquid crystals. However, there is a class of liquid crystals for which $k > 0$. In fact, there is a temperature dependence of this parameter for polymer liquid crystals [12,13], and the ratio K_{11}/K_{33} can be written as [14]

$$\frac{K_{11}}{K_{33}} \propto \left(\frac{U_h}{K_B T}\right)^{7/4} e^{U_h/K_B T}. \tag{7}$$

Eq. (7) shows that, for these materials, k decreases with increasing temperature.

3. Critical phenomenon

The dependence of the director on the relevant parameters in the vicinity of the transition can be analyzed by investigating ψ_M . As a matter of fact, if $\psi_M = 0$, the system is non-distorted; if $\psi_M \neq 0$, the system is distorted. For this reason, ψ_M can be assumed as an order parameter as is done for liquid crystals in common geometries [15]. In order to investigate the critical behavior, let us consider the sample in the x -space, where it has (dimensionless) thickness $d = \ln \rho$. Therefore, there is a critical thickness given by d_c (see Eq. (6)). The complete expression for the free energy can be rewritten as

$$F[\psi(x)] = \int_0^d \left\{ \left(k - \delta^2 - k \left[\frac{d\psi}{dx} \right]^2 \right) \cos^2 \psi + (k + 1) \left[\frac{d\psi}{dx} \right]^2 \right\} dx, \tag{8}$$

with $\delta^2 = (V/V_c)^2/[k - (\pi/d)^2]$, where a constant term was omitted. Minimizing the free energy, after some calculations, we obtain the following differential equation:

$$\frac{1}{2} \sin 2\psi \left\{ k - \delta + k \left[\frac{d\psi}{dx} \right]^2 \right\} + (1 + k \sin^2 \psi) \frac{d^2\psi}{dx^2} = 0. \tag{9}$$

By using the relations $\psi(x/2) = \psi_M$ and $(d\psi/dx)_{x=d/2} = 0$, it is possible to obtain a first integral of Eq. (9), and the function $\psi(x)$ can be given by

$$\int_0^{\psi(x)} \sqrt{\frac{1 + k \sin^2 \zeta}{\sin^2 \psi_M - \sin^2 \zeta}} d\zeta = xh, \tag{10}$$

which, by means of the change of variables $\sin \varphi = \sin \zeta / \sin \psi_M$, and integrating from $x = 0$ to $x = d/2$, can be put in the form

$$\int_0^{\pi/2} \sqrt{\frac{1 + k \sin^2 \psi_M \sin^2 \varphi}{1 - \sin^2 \psi_M \sin^2 \varphi}} d\varphi = \frac{d}{2} h. \tag{11}$$

This equation gives ψ_M as a function of the relevant parameters of the transition. In the limit $\psi_M \rightarrow 0$, it is possible to obtain the critical potential found in Section 2.

To investigate the behavior of the order parameter in the vicinity of the transition, we expand Eq. (11) in series of ψ_M . By neglecting higher order terms, we obtain

$$\frac{\pi}{2} + \frac{\pi}{8} (1 + k) \psi_M^2 = \frac{d}{2} h, \tag{12}$$

giving

$$\psi_M = \frac{2}{\sqrt{k+1}} \sqrt{\frac{d}{\pi} \sqrt{k - \frac{V^2}{V_c^2} \left(k - \frac{\pi^2}{d^2} \right)} - 1}. \tag{13}$$

One concludes that the behavior of ψ_M , when $V \rightarrow V_c$, can be well described by

$$\psi_M \propto (V - V_c)^{1/2}. \tag{14}$$

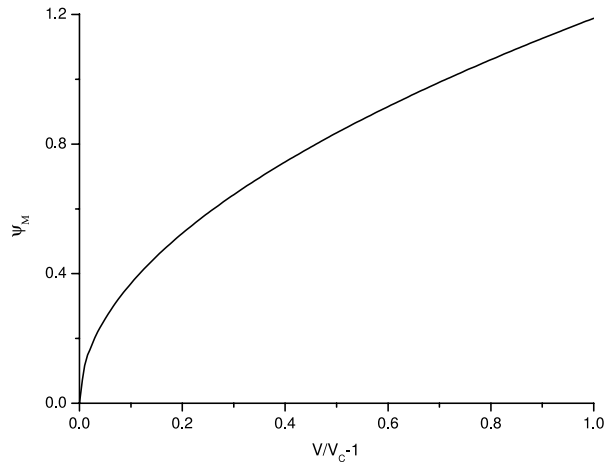


Fig. 2. $\psi_M \times (V/V_c - 1)$ for $k = 1.5$ and $d = 1$. It is possible to observe that the order parameter increases with $(V/V_c - 1)^{1/2}$.

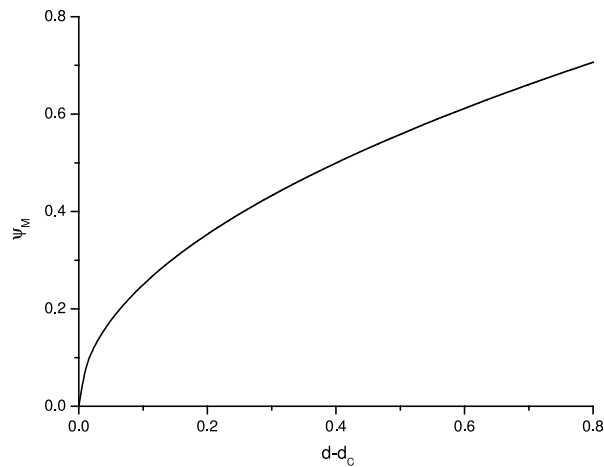


Fig. 3. Dependence of ψ_M on $(d - d_c)$ for $k = 1.5$.

This fact is illustrated in Fig. 2. The critical exponent is the same as the one found in mean field theory and accounts for the usual behavior of the order parameter in the Fréedericksz transition in a hybrid cell when a planar geometry is considered [15].

As shown in the in Section 2, there is a Fréedericksz transition even in the absence of the external (electrical or magnetic) field. By putting $V = 0$ into Eq. (13), and using the definition of the critical thickness, the behavior of ψ_M can be expressed as

$$\psi_M = f(k) (d - d_c)^\beta, \tag{15}$$

with

$$\beta = 1/2 \quad \text{and} \quad f(k) = 2 \sqrt[4]{\frac{k}{\pi^2(k+1)^2}}.$$

In Eq. (15), the thickness of the sample plays a role similar to the one of the temperature in usual second-order phase transitions [16]. It follows that the critical exponent is the classical one, i.e., $\beta = 1/2$ (Fig. 3). The function $f(k)$ is not a monotonic function as shown in Fig. 4. One observes that $f(k)$ has a maximum in $k = 1$, where $f(1) = \sqrt{\pi}/2 \approx 0.8$. This suggests that if $d \gtrsim d_c$, the director profile can be approximately given by (4), with ψ_M given by (13) for V ranging from zero to a value slight above V_c .

Eq. (7) shows that the parameter k has a temperature dependence. In this way, k can be faced as the parameter that induces a transition in the ordering of the director. For a fixed thickness d , one easily finds $k_c = \pi^2/d^2$. Substitution of this result into Eq. (13), for $V = 0$, yields

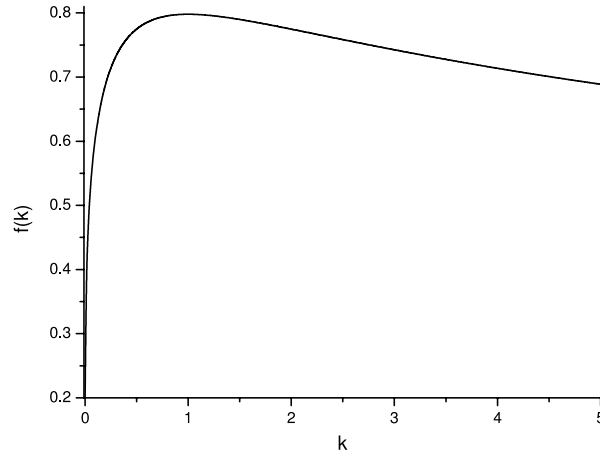


Fig. 4. Behavior of $f(k)$. The maximum value of $f < 1$. The profile of the director is well approximated by the configuration obtained in the small distortions regime.

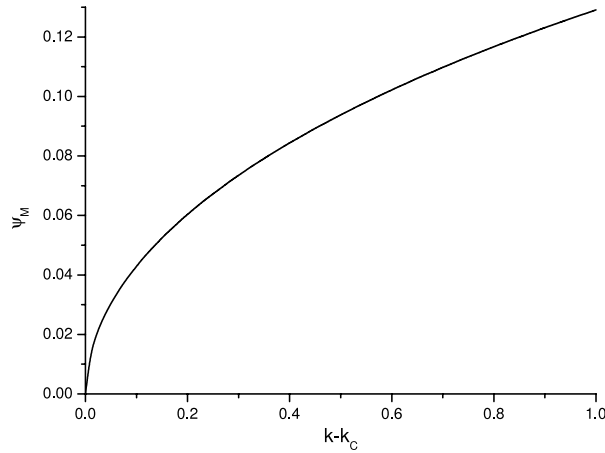


Fig. 5. Dependence of ψ_M on $(k - k_c)$ for $d = 1$.

$$\psi_M = \frac{2}{\sqrt{\frac{\pi^2}{d^2} + 1 + (k - k_c)}} \sqrt{\frac{d}{\pi} \sqrt{\frac{\pi^2}{d^2} + (k - k_c)} - 1} \tag{16}$$

which is not in the usual asymptotic form in the vicinity of the transition. However, if d is small, it can be approximated as

$$\psi_M \approx \frac{\sqrt{2}d^2}{\pi^2} (k - k_c)^{1/2}. \tag{17}$$

Numerical simulations show that this approximation works well for $d \lesssim 1.5$, where the error is about 2% for $k - k_c \sim 1$. For this range of thicknesses, the critical exponent is also 1/2 (Fig. 5). Since $d = \ln \rho$, the limiting value $d \approx 1.5$ is equivalent to $\rho = e^{1.5} \approx 4.5$, indicating that the behavior predicted by Eq. (17) works well for a significantly large range of separations between the cylindrical shells containing the sample.

4. Concluding remarks

In this work, we have investigated the influence of an electrical field on a sample of liquid crystal confined in a complex geometry, i.e., a nematic medium confined between two concentric cylinders. In the analysis presented here, different from the previous analysis of the problem, the director is free to move only in the polar plane, i.e., in the plane normal to the cylinders' long axes. The threshold potential for Fréedericksz was determined and also a Fréedericksz transition in the absence of the external field was predicted. By solving the exact differential equation, we have investigated the behavior of the system in the vicinity of the transition and, in this manner, we have obtained the order parameter critical exponent in two scenarios: In the presence and in the absence of an external field. In both cases, it is possible to consider that the

“intensive variable” driving the transition is the thickness of the sample and, by analogy to what happens in a usual second-order phase transition, the critical exponent is the classical one, i.e., $\beta = 1/2$. In the case in which the elastic constants are temperature dependent and the thickness is small, the same critical exponent is found. In this situation, the driving parameter for the transition is the elastic anisotropy k . In our analysis, the elastic anisotropy has to be such that $k \geq 0$ in order to observe a Fréedericksz transition in the absence of the external field. This implies that $K_{11} > K_{33}$, which is not true for a large number of usual liquid crystal, but is true polymer liquid crystals. However, in the geometry used by Ref. [4], the transition in the absence of the external field is possible for all the class of liquid crystals. Anyway, the critical exponents are expected to be the same for both geometries. In a broad scenario, these results show us that nontrivial consequences may occur when complex fluids are subject to non-planar geometries.

Acknowledgements

This work was partially supported by Brazilian Agencies Capes, CNPq (Instituto Nacional de Ciência e Tecnologia de Fluidos Complexos) and Fundação Araucária.

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