Connection between Dirichlet distributions and a scale-invariant probabilistic model based on Leibniz-like pyramids

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Connection between Dirichlet distributions and a scale-invariant probabilistic model based on Leibniz-like pyramids

A Rodríguez[1] **and C Tsallis**[2,3]

[1] Departamento de Matemática Aplicada a la Ingeniería Aeroespacial, Universidad Politécnica de Madrid, Pza. Cardenal Cisneros s/n, 28040 Madrid, Spain
[2] Centro Brasileiro de Pesquisas Físicas and Instituto Nacional de Ciência e Tecnologia de Sistemas Complexos, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil
[3] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA
E-mail: antonio.rodriguezm@upm.es and tsallis@cbpf.br

Received 4 November 2014
Accepted for publication 30 November 2014
Published 22 December 2014

Online at stacks.iop.org/JSTAT/2014/P12027
doi:10.1088/1742-5468/2014/12/P12027

Abstract. We show that the $N \to \infty$ limiting probability distributions of a recently introduced family of d-dimensional scale-invariant probabilistic models based on Leibniz-like $(d+1)$-dimensional hyperpyramids (Rodríguez and Tsallis 2012 *J. Math. Phys.* **53** 023302) are given by Dirichlet distributions for $d = 1, 2, \ldots$. It was formerly proved by Rodríguez *et al* that, for the one-dimensional case ($d = 1$), the corresponding limiting distributions are q-Gaussians ($\propto e_q^{-\beta x^2}$, with $e_1^{-\beta x^2} = e^{-\beta x^2}$). The Dirichlet distributions generalize the so-called Beta distributions to higher dimensions. Consistently, we make a connection between one-dimensional q-Gaussians and Beta distributions via a linear transformation. In addition, we discuss the probabilistically admissible region of parameters q and β defining a normalizable q-Gaussian, focusing particularly on the possibility of having both bell-shaped and U-shaped q-Gaussians, the latter corresponding, in an appropriate physical interpretation, to negative temperatures.

Keywords: rigorous results in statistical mechanics

1742-5468/14/P12027+15$33.00

Contents

1. Introduction

Within the framework of q-statistics [1,2], q-Gaussians (to be introduced in detail later on, in section 2) emerge when optimizing under appropriate constraints [3–5] the nonadditive entropy [1] $S_q = k\frac{1-\int \mathrm{d}x[p(x)^q]}{q-1}$, $q \in \mathbb{R}$, similarly to the manner through which Gaussians emerge from the Boltzmann–Gibbs (BG) additive entropy ($S_1 \equiv S_{\mathrm{BG}} = -k\int \mathrm{d}x \ln(p(x))$) in the standard BG statistical mechanics. Physically speaking, q-Gaussian distributions naturally extend Gaussian distributions for nonergodic and other strongly correlated systems. In fact, q-Gaussians have been found in analytical, numerical, experimental and observational studies of anomalous diffusion [6], granular matter [7], long-range-interacting many-body classical Hamiltonians [8], solar wind [9], cold atoms in optical lattices [10–12], over damped motion of interacting vortices in type-II superconductors [13], motion of *Hydra viridissima* and other micro-organisms [14, 15], plasma physics [16–19], trapped ions [20], among others.

The relationship between q-Gaussianity (having q-Gaussians as attractors of the probability distributions in the thermodynamic limit), extensivity (whose associated entropic functional satisfies $S(N) \propto N$ for $N \gg 1$) and scale-invariance (a specific way to introduce correlations in the system to be described in section 5) has raised much attention in recent years. Since [21], a number of probabilistic models have been introduced with this purpose [22–26]. Nevertheless, the search for a statistical model which provides both S_q entropy with $q \neq 1$ and q-Gaussians as attractors has been up to now unsuccessful.

In such a context, little attention has been paid to multivariate models. In a recent paper [22] we introduced a family of d-dimensional scale invariant probabilistic models corresponding to the sum of N $(d+1)$-valued variables (binary variables for $d=1$, ternary variables for $d=2$ and so on), based on a family of $(d+1)$-dimensional Leibniz-like hyperpyramids. We showed that the corresponding distributions in the thermodynamic limit were q-Gaussians only in the one-dimensional (corresponding to binary variables) case. In the present contribution we show that the family of attractors are the so called Dirichlet distributions (to be described in section 4) for any value of d and present a detailed demonstration for $d=1$, 2 and 3 in section 5. The one-dimensional case, which will allow us to establish a connection between q-Gaussians and symmetric Beta distributions, up to now unnoticed, will be presented in section 2. We will also comment on the relationship between parameters q and β defining a q-Gaussian in section 3.

2. *q*-Gaussians and beta distributions

One-dimensional q-Gaussians are defined as follows:

$$G_q(\beta;x) = C_{q,\beta}\mathrm{e}_q^{-\beta x^2}; \quad x \in D_q \tag{1}$$

where $\beta \in \mathbb{R}$, $C_{q,\beta}^{-1} = \int_{D_q} \mathrm{e}_q^{-\beta x^2}\mathrm{d}x$, and we have made use of the q-*exponential* function $\mathrm{e}_q^x \equiv [1+(1-q)x]_+^{\frac{1}{1-q}}$ (the symbol $[z]_+$ indicates $[z]_+ = z$ if $z \geqslant 0$ and 0 otherwise) with $\mathrm{e}_1^x = \mathrm{e}^x$ (so a Gaussian distribution may be interpreted as a q-Gaussian distribution with $q=1$).

q-Gaussians are normalizable probability distributions whenever $q<3$ for $\beta>0$ and $q>2$ for $\beta<0$ (the negative β case has never been explicitly studied in the literature before). The support is $D_q = \mathbb{R}$ if $q \in [1,3)$, $\beta>0$ and the bounded interval $D_q = \{x \;/\; |x| < 1/\sqrt{\beta(1-q)}\,\}$ for $q<1$, $\beta>0$ and $q>2$, $\beta<0$ (the boundary of the interval is reached only in the $\beta>0$ case, thus having a compact support). The expression for the normalization constant is given by

$$C_{q,\beta} = \begin{cases} \dfrac{\sqrt{\beta(1-q)}}{B\left(\frac{2-q}{1-q},\frac{1}{2}\right)}; & \beta>0,\ q<1 \quad \text{and} \quad \beta<0,\ q>2 \\[2ex] \dfrac{\sqrt{\beta(q-1)}}{B\left(\frac{3-q}{2(q-1)},\frac{1}{2}\right)}; & \beta>0,\ 1<q<3 \end{cases} \tag{2}$$

where $B(x,y)$ stands for the Beta function with $\lim_{q\to 1} C_{q,\beta>0} = \frac{\sqrt{\beta}}{\sqrt{\pi}}$, as expected for the Gaussian distribution.

In this section, we will be mainly concerned with the bounded support case. We shall do the linear change of variable [27]

$$x = \frac{1}{\sqrt{\beta(1-q)}}(2y-1) \tag{3}$$

which transforms the bounded support D_q into the interval $(0,1)$. The probability distribution function for the new variable is

$$f_q(y) \equiv G_q(\beta;x(y))\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2C_{q,\beta}4^{\frac{1}{1-q}}}{\sqrt{\beta(1-q)}}(y(1-y))^{\frac{1}{1-q}}; \quad y \in (0,1) \tag{4}$$

J. Stat. Mech. (2014) P12027

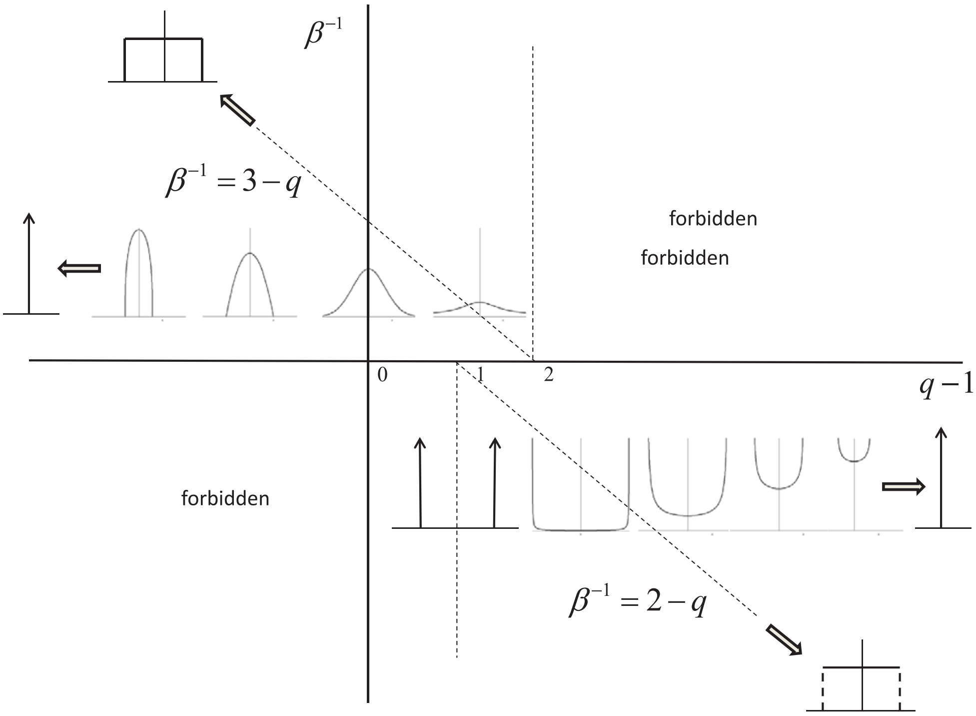

Figure 1. q-Gaussians, $e_q^{-\beta x^2}$, in the q–β plane with vertical axis β^{-1} and horizontal axis $q-1$. Shadowed regions correspond to forbidden values of (q, β). The support of the uniform distribution in the upper (lower) β^{-1} semiplane is compact (open). See text for the limiting distributions in the diagonal lines.

Making use of the expression for $C_{q,\beta}$ given in (2) and the duplication formula of the Gamma function, the normalization constant in (4) can be reexpressed and we finally get

$$f_q(y) = \frac{1}{B\left(\frac{2-q}{1-q}, \frac{2-q}{1-q}\right)} y^{\frac{1}{1-q}} (1-y)^{\frac{1}{1-q}}; \quad y \in (0,1) \tag{5}$$

which is a symmetric Beta distribution ($f(y; \alpha_1, \alpha_2) = \frac{1}{B(\alpha_1,\alpha_2)} y^{\alpha_1 - 1}(1-y) y^{\alpha_2 - 1}$, $y \in (0,1)$, $\alpha_1, \alpha_2 > 0$), with $\alpha_1 - 1 = \alpha_2 - 1 = \frac{1}{1-q}$. We thus conclude that under change (3), a one-dimensional q-Gaussian with bounded support transforms into a symmetric Beta distribution function with parameters $\alpha_1 = \alpha_2 = \frac{2-q}{1-q} > 0$. As we will show in section 5 this is *not* the case in higher dimensions.

3. The q–β plane

Since β^{-1} plays the role of an effective temperature, the $\beta < 0$ values yield q-Gaussians that are to be associated with systems at negative temperatures. As the state of a system is determined by the values of q and T, any allowed point in a q–β plane will be in correspondence with a different q-Gaussian. Figure 1 shows such a plane and how q-Gaussians look like depending on their location in it. To start with, any q-Gaussian with $\beta > 0$ points downwards (being concave for $q < 0$ and having two symmetric inflection points for $q > 0$) while any q-Gaussian with $\beta < 0$ is convex [28]. In the upper half-plane,

J. Stat. Mech. (2014) P12027

q-Gaussians for a constant value of $\beta > 0$ are shown for decreasing values of $q < 3$ from right to left. In the vertical axis, corresponding to $q = 1$, the Gaussian is shown. On its right, a q-Gaussian with $1 < q < 3$ is depicted, whose support is the whole real axis. As q further increases q-Gaussians turns flatter up to the limit $q \to 3^-$ where distributions are no longer normalizable, with $\lim_{q\to 3^-} C_{q,\beta>0} = 0$. On the left top quadrant, q-Gaussians for $q < 1$ and a constant value of $\beta > 0$ are shown, all of them with bounded support D_q with $\lim_{q\to -\infty} D_q = 0$, while the area is preserved. Thus a Dirac delta function is obtained in the $q \to -\infty$ limit provided β is kept constant. Nevertheless, the same limit can be taken following different paths given by the graph of any function $\beta(q)$ on the plane. In [3] the choice $\beta(q) = \frac{1}{3-q}$ was made. Thus, the support is $D_q = \{x \;/\; |x| \leqslant \sqrt{(3-q)/(1-q)}\}$ with $\lim_{q\to -\infty} D_q = [-1, 1]$ so in the limit a uniform distribution is obtained instead, as shown in figure 1. The same limiting distribution would be obtained following any $\beta(q)$ curve provided $\beta(q) = O((1-q)^{-1})$.

In the lower plane, convex q-Gaussians for a constant value of $\beta < 0$ are shown for increasing values of $q \geqslant 2$ from left to right. All of them have a bounded support D_q with $\lim_{q\to\infty} D_q = 0$. Also, $\lim_{q\to\infty} G_q(\beta < 0; 0) = \infty$. Thus, we obtain again a Dirac delta distribution in the $q \to \infty$ limit when $\beta < 0$ is kept constant. In the opposite $q \to 2^+$ limit, with support $D_2 = [-1/\sqrt{-\beta}, 1/\sqrt{-\beta}]$ and $\lim_{q\to 2^+} C_{q,\beta<0} = 0$, a double peaked delta distribution is obtained as can be seen by doing the change (3) and considering the corresponding limit of the beta distribution (5). In analogy with the path followed in the upper half plane, taking $\beta(q) = \frac{1}{2-q}$ we obtain $D_q = \{x \;/\; |x| \leqslant \sqrt{(2-q)/(1-q)}\}$ with $\lim_{q\to\infty} D_q = [-1, 1]$. Thus, the support remains finite while the area is preserved, so we recover the uniform distribution. Again, the same limit is obtained for any path with $\beta(q) = O((1-q)^{-1})$.

4. Dirichlet distributions

The so called d-dimensional Dirichlet distribution is defined in the form [29]

$$f(x_1, \ldots, x_d; \alpha_1, \ldots, \alpha_{d+1}) = \frac{\Gamma(\sum_{i=1}^{d+1} \alpha_i)}{\prod_{i=1}^{d+1} \Gamma(\alpha_i)} x_1^{\alpha_1 - 1} \cdots x_d^{\alpha_d - 1} (1 - x_1 - \cdots - x_d)^{\alpha_{d+1} - 1} \tag{6}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{d+1} > 0$, and is defined in the simplex

$$D = \left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \;/\; x_1, x_2, \ldots, x_d > 0,\; x_1 + x_2 + \cdots + x_d < 1\right\} \tag{7}$$

For $d = 1$ distribution (6) reduces to the Beta distribution. Dirichlet distributions then generalize Beta distribution to higher dimensions. In addition, it may be shown that one-dimensional marginal distributions of Dirichlet distributions (6) $\vec{X} \sim \mathrm{Dir}(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})$ are Beta distributions in the form $X_i \sim \mathrm{Beta}(\alpha_i, \sum_{j\neq i} \alpha_j)$. We shall use this property in section 5.

Figure 2 shows bidimensional Dirichlet distributions for the specified values of parameters α_1, α_2 and α_3. They present a maximum (minimum) at $(\frac{\alpha_1 - 1}{\alpha_1+\alpha_2+\alpha_3-3}, \frac{\alpha_2 - 1}{\alpha_1+\alpha_2+\alpha_3-3})$ whenever $\alpha_1, \alpha_2, \alpha_3 > 1$ $(\alpha_1, \alpha_2, \alpha_3 < 1)$. If two of the parameters equal one a plane surface is obtained, including the uniform distribution for $\alpha_1 = \alpha_2 = \alpha_3 = 1$. In the following we will be mainly interested in the symmetric case $\alpha_1 = \alpha_2 = \alpha_3$.

J. Stat. Mech. (2014) P12027

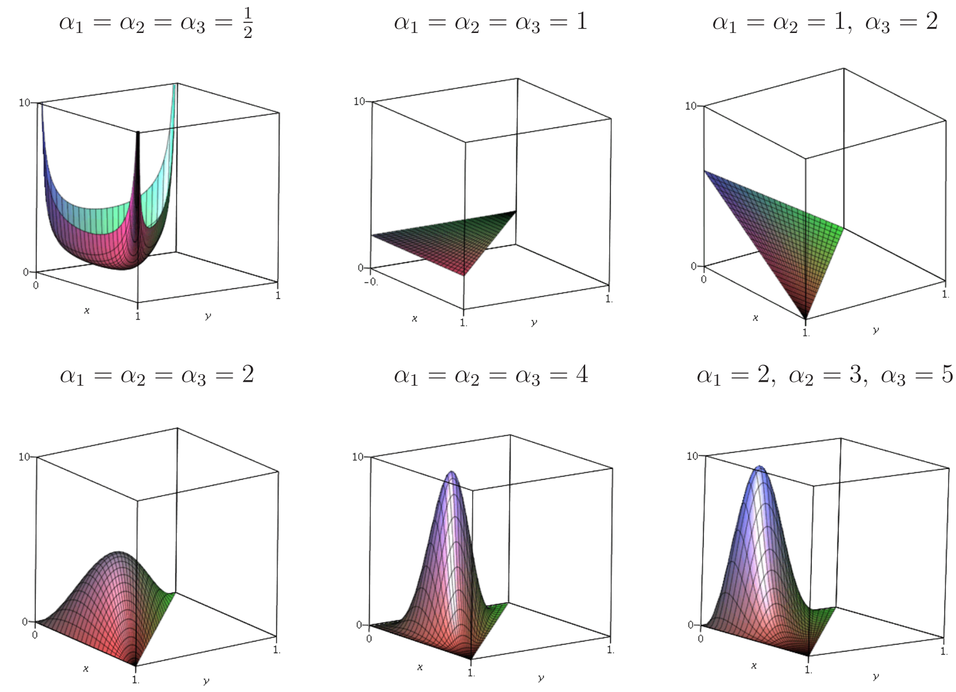

Figure 2. Bidimensional Dirichlet distributions for different values of parameters α_1, α_2 and α_3.

In turn the d-dimensional q-Gaussian distribution reads

$$G_q(\beta, \Sigma; x_1, x_2, \ldots, x_d) = C_{q,d}\, \mathrm{e}_q^{-\beta \vec{x}^T \Sigma \vec{x}}; \quad \vec{x}^T = (x_1, x_2, \ldots, x_d) \in \Omega_q \tag{8}$$

where $\beta \in \mathbb{R}$, Σ is a positive definite matrix, $C_{q,d}^{-1} = \int_{\Omega_q} \mathrm{e}_q^{-\beta \vec{x}^T \Sigma \vec{x}} \mathrm{d}x_1 \cdots \mathrm{d}x_d$, and the support is $\Omega_q = \mathbb{R}^d$, if $q \geqslant 1$, $\beta > 0$ and the solid hyperellipsoid $\Omega_q = \{(x_1, x_2, \ldots, x_d) \;/\; \vec{x}^T \Sigma\, \vec{x} < \frac{1}{\beta(1-q)}\}$, for $q < 1$, $\beta > 0$ and $q > 2$, $\beta < 0$ (again, the border of the support is reached only in the $\beta > 0$ case). It can be shown that distributions (8) are normalizable for $q < 1 + \frac{2}{d}$ when $\beta > 0$ and for $q > 2$ when $\beta < 0$ having all its moments defined in the bounded support case and moments up to the m-th one defined only if $q < 1 + \frac{2}{m+d}$ in the $\Omega_q = \mathbb{R}^d$ case.

Some representative bidimensional q-Gaussians are plotted in figure 3. They have a minimum (maximum) at the origin for $\beta < 0$ ($\beta > 0$). For $\beta < 0$ the height of the minimum increases and the radius of the circular support decreases for increasing values of $q > 2$ whereas for $\beta > 0$ the height of the maximum decreases and the radius of the support increases for increasing values of $q < 2$.

Contrary to the one-dimensional case, d-dimensional q-Gaussians with bounded support and $d \geqslant 2$ are *not* able to be transformed into Dirichlet distributions via a change of variables. Let us focus on the simplest $d = 2$ case for which the normalization constant in (8) takes the simple expression $C_{q,2} = \frac{\beta(2-q)|\Sigma|^{1/2}}{\pi}$. If we even further take $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$, the corresponding symmetric Dirichlet distribution reads

$$f(x, y; \alpha) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)^3} (xy(1 - x - y))^{\alpha - 1} \tag{9}$$

J. Stat. Mech. (2014) P12027

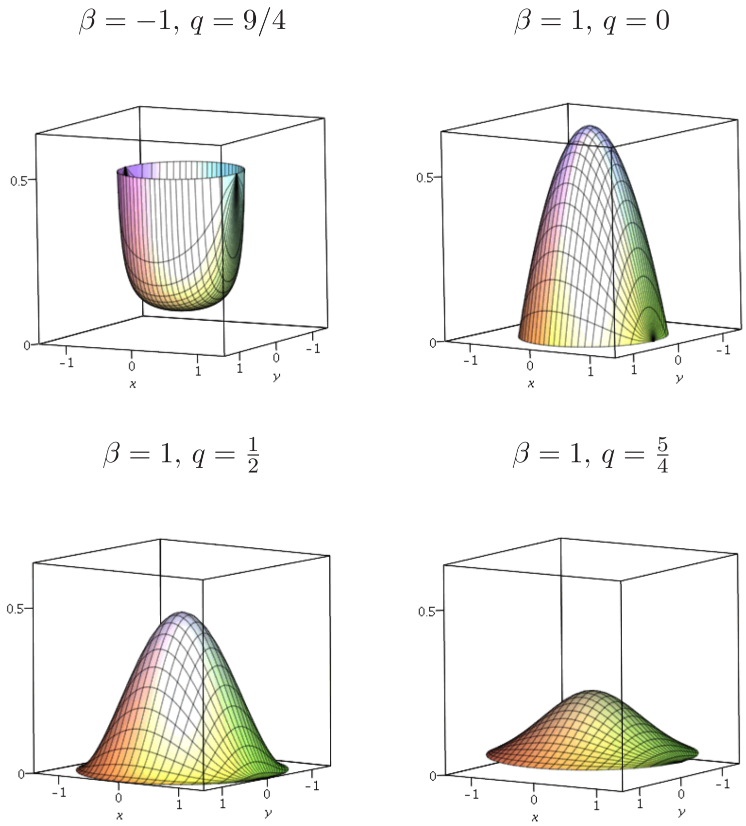

Figure 3. Plot of some bidimensional q-Gaussians (8) for typical values of q and β.

and is defined on the triangle $x > 0, y > 0, x + y < 1$, while the corresponding bidimensional q-Gaussian, once diagonalized (we shall take $\Sigma = I$ for simplicity) reads

$$G_q(x, y) = \frac{\beta(2-q)}{\pi}[1 - (1-q)\beta(x^2 + y^2)]_+^{\frac{1}{1-q}} \tag{10}$$

being defined on the circle $x^2 + y^2 < \frac{1}{\beta(1-q)}$.

Distribution (10) has cylindric symmetry while distribution (9) has only symmetry of rotation of angle $2\pi/3$ about the barycenter $(\frac{1}{3}, \frac{1}{3})$ of the triangle support. Even in the case $\frac{1}{1-q} = \alpha - 1$, it is *not* possible to convert (10) into (9) via a linear transformation. Whether there exist highly non trivial changes that transform Dirichlet distributions into bounded support q-Gaussians (or equivalently simplexes in hyperellipsoids) is out of the scope of this paper.

5. Limiting probability distributions of a family of scale-invariant models

In [22], we introduced a one-parameter family of discrete, scale-invariant (in a sense to be made explicit below) probabilistic models describing a variable $\vec{X} = \vec{X}_1 + \vec{X}_2 + \cdots + \vec{X}_N$, that is the sum of N $(d+1)$-valued d-dimensional variables (thus $\vec{X}$ can be associated

J. Stat. Mech. (2014) P12027

with the throwing of N $(d+1)$-sided dice) with a probability function in the form

$$p^{(\nu)}_{N,n_1,n_2,\ldots,n_d} = \binom{N}{n_1, n_2, \ldots, n_d} r^{(\nu)}_{N,n_1,n_2,\ldots,n_d} \tag{11}$$

where $\nu > 0$, $p^{(\nu)}_{N,n_1,n_2,\ldots,n_d} \equiv P(X_1 = n_1, \ldots, X_d = n_d)$, with n_i nonnegative integers for $i = 1, \ldots, d$, and $n_1 + n_2 + \cdots + n_d \leqslant N$. The multinomial coefficients in (11) stand for the degeneracy arising from the exchangeability of variables $\vec{X}_i$. Thus, the sample space with d^N events splits into $\frac{(N+1)(N+2)\ldots(N+d)}{d!}$ regions, such that the $\binom{N}{n_1, n_2, \ldots, n_d}$ events belonging to each of them take all the same probability given by the coefficients in (11) defined as

$$r^{(\nu)}_{N,n_1,n_2,\ldots,n_d} = \frac{\Gamma((d+1)\nu)}{\Gamma(\nu)^{d+1}} \frac{\Gamma(N - \sum_{i=1}^{d} n_i + \nu)\Gamma(n_1+\nu)\Gamma(n_2+\nu)\cdots\Gamma(n_d+\nu)}{\Gamma(N+(d+1)\nu)}. \tag{12}$$

Coefficients (12) may be displayed in a hyperpyramid (a triangle for $d = 1$, a pyramid for $d = 2$ and so on, see [22] for details) in the same fashion as the multinomial coefficients do, and due to the aforementioned scale-invariance satisfy certain relations among them.

We say that a probabilistic system consisting of N random variables $x_1, x_2, \ldots, x_N$ with joint probability distribution $p_N(x_1, x_2, \ldots, x_N)$ is scale-invariant when the functional form of the $N-1$ dimensional marginal distribution of a $(N-1)$-variables subset $\tilde{p}_{N-1}(x_1, x_2, \ldots, x_{N-1}) = \int p_N(x_1, x_2, \ldots, x_N)\mathrm{d}x_N$, coincides with that of the joint probability distribution of the $N-1$ subset $p_{N-1}(x_1, x_2, \ldots, x_{N-1})$. This condition, trivially fulfilled in the case of independent variables, involves, in the absence of independence, the presence of longe-range correlations. In our model, the scale-invariance condition is traduced in the form of restrictions on the coefficients (12), that for the simplest $d = 1$ case reduce to the so called Leibniz rule

$$r^{(\nu)}_{N,n} + r^{(\nu)}_{N,n+1} = r^{(\nu)}_{N-1,n} \tag{13}$$

while for $d \geqslant 2$ can be cast in the form $(\vec{n} \equiv (n_1, n_2, \ldots, n_d))$:

$$r^{(\nu)}_{N,\vec{n}} + r^{(\nu)}_{N,\vec{n}+\vec{\varepsilon}_1} + \cdots + r^{(\nu)}_{N,\vec{n}+\vec{\varepsilon}_d} = r^{(\nu)}_{N-1,\vec{n}+\vec{\varepsilon}_d} \tag{14}$$

where $\vec{\varepsilon}_1 = \vec{e}_1 - \vec{e}_2$, $\vec{\varepsilon}_i = \vec{e}_{i+1} - \vec{e}_2$, for $i = 2, \ldots, d-1$, and $\vec{\varepsilon}_d = -\vec{e}_2$; with $\vec{e}_i$ for $i = 1, \ldots, d$ being the vectors of the canonical basis of $\mathbb{R}^d$.

Our main claim here is the following: *The family of probability distributions (11) characterized by parameter $\nu > 0$ has, in the thermodynamic limit, symmetric Dirichlet distributions as attractors, with $\alpha_1 = \cdots = \alpha_{d+1} = \nu$.*

We shall develop our proof separately for increasing values of d.

5.1. $d = 1$

In this case coefficients (12) reduce to

$$r^{(\nu)}_{N,n} = \frac{B(N-n+\nu, n+\nu)}{B(\nu,\nu)}; \quad \nu > 0 \tag{15}$$

with $n = 0, 1, \ldots, N$, and satisfy relation (13). Coefficients (15) may be displayed in a triangle in the plane as is the case of the Pascal coefficients. The associated probabilities are

$$p^{(\nu)}_{N,n} = \binom{N}{n} r^{(\nu)}_{N,n} \tag{16}$$

J. Stat. Mech. (2014) P12027

We shall now generalize to any $\nu > 0$ the demonstration on the attractor of distribution (16) given only for positive integer values of ν in [23].

By doing the change $t = 1 - \mathrm{e}^{-u}$ in the definition of the Beta function in (15) one gets

$$B(\nu,\nu) r_{N,n}^{(\nu)} = \int_0^1 t^{N-n+\nu-1}(1-t)^{n+\nu-1}\mathrm{d}t \tag{17a}$$

$$= \int_0^\infty (1-\mathrm{e}^{-u})^{N-n+\nu-1}\mathrm{e}^{-u(n+\nu)}\mathrm{d}u \tag{17b}$$

$$= \int_0^\infty (1-\mathrm{e}^{-u})^{\nu-1}\mathrm{e}^{-\nu u}\mathrm{e}^{Nf(u)}\mathrm{d}u \tag{17c}$$

where we have defined $f(u) = (1-x)\ln(1-\mathrm{e}^{-u}) - xu$, with $x = \frac{n}{N}$, $0 \leqslant x \leqslant 1$. Applying now the Laplace method to integral (17c) where f has a maximum at $u^\star = -\ln x$, with $f(u^\star) = (1-x)\ln(1-x) + x\ln x$, and $f''(u^\star) = -\frac{x}{1-x}$, after some manipulations one gets

$$B(\nu,\nu) r_{N,n}^{(\nu)} \simeq \sqrt{\frac{2\pi}{N}} \times x^\nu(1-x)^{\nu-1} \times x^{Nx-\frac{1}{2}}(1-x)^{N(1-x)+\frac{1}{2}}. \tag{18}$$

Applying now Stirling approximation to the binomial coefficients one easily gets

$$\binom{N}{n} \simeq \frac{1}{\sqrt{2\pi N}} \times \frac{1}{x^{Nx+\frac{1}{2}}(1-x)^{N(1-x)+\frac{1}{2}}}. \tag{19}$$

Finally, as the change $Nx = n$ transforms probability distribution (16) in $\mathcal{P}_N^{(\nu)}(x) = Np_{N,n}^{(\nu)}$, taking into account (18) and (19) yields

$$\lim_{N\to\infty} \mathcal{P}_N^{(\nu)}(x) \equiv \mathcal{P}^{(\nu)}(x) = \frac{x^{\nu-1}(1-x)^{\nu-1}}{B(\nu,\nu)}; \quad x \in (0,1) \tag{20}$$

which is a symmetric Beta distribution with parameters $\alpha_1 = \alpha_2 = \nu > 0$, as above claimed. Next, by only doing the inverse change of (3)

$$x = \frac{1}{2}\left(\sqrt{\beta(1-q)}\, y + 1\right). \tag{21}$$

Beta distribution (20) transforms into a q-Gaussian distribution with $q = \frac{\nu-2}{\nu-1}$, as seen in section 2. For $\nu \in (0,1)$ we get $q > 2$, so $\beta < 0$ and the q-Gaussian is convex, whereas for $\nu > 1$ we get $q < 1$ and $\beta > 0$ so q-Gaussians pointing downwards are obtained. As shown in [27], by means of a more complicated nonlinear change of variable that we will not show here, it is also possible to obtain q-Gaussians with $q > 1$ and $\beta > 0$ out of distribution (16).

The left panel of figure 4 shows normalized probability distributions (16) $Np_{N,n}^{(\nu)}$ versus n/N for different values of ν and $N = 100$ (dotted lines) together with their corresponding Beta distributions (20) (solid lines). The right panel of figure 4 shows a detail of the distributions $Np_{N,n}^{(\nu)}$ for $\nu = 5$ and increasing values of N. The convergence to the corresponding Beta distribution $\mathcal{P}^{(5)}(x)$ is apparent.

5.2. $\boldsymbol{d} = 2$

Let us now turn to the bidimensional case where coefficients (12) reduce to

$$r_{N,n,m}^{(\nu)} = \frac{B(N-n-m+\nu, n+m+2\nu)B(n+\nu, m+\nu)}{B(\nu,\nu)B(\nu,2\nu)}; \quad \nu > 0 \tag{22}$$

J. Stat. Mech. (2014) P12027

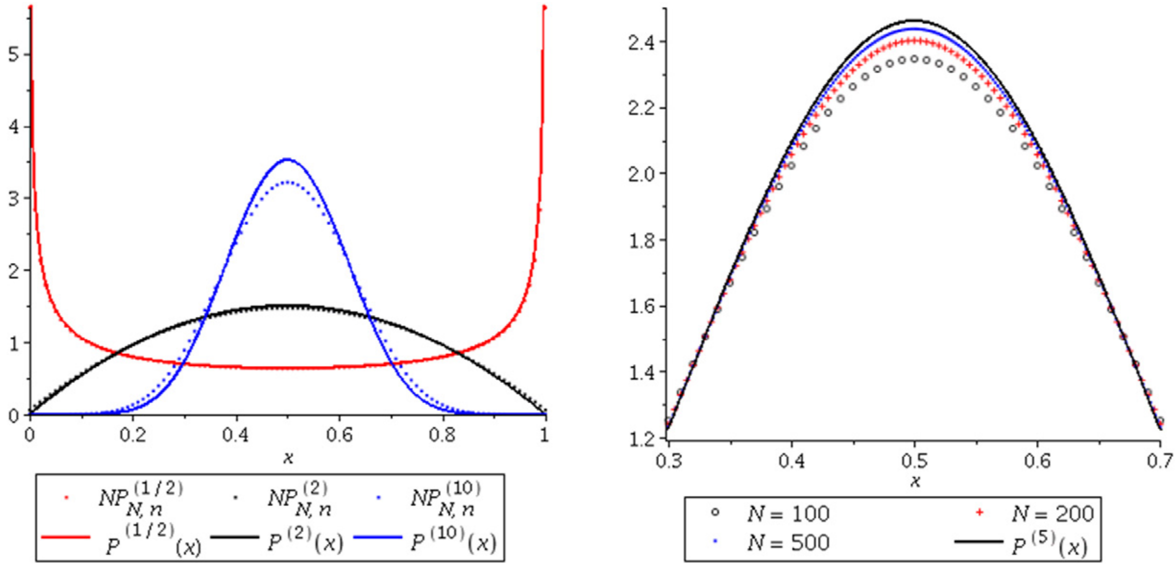

Figure 4. (Left) Distributions $Np^{(\nu)}_{N,n}$ versus $x = n/N$ for $\nu = 1/2$, 2 and 10 with $N = 100$ together with their corresponding Beta distributions $\mathcal{P}^{(\nu)}(x)$ (in solid line). (Right) Detail of the top part of the distributions $Np^{(\nu)}_{N,n}$ for $\nu = 5$ and $N = 100$, 200 and 500 compared with the Beta distribution $\mathcal{P}^{(5)}(x)$.

with $n, m \geqslant 0$, $n + m \leqslant N$. Coefficients (22) obey the corresponding relation (14), which for $d = 2$ yields

$$r^{(\nu)}_{N,n,m} + r^{(\nu)}_{N,n+1,m-1} + r^{(\nu)}_{N,n,m-1} = r^{(\nu)}_{N-1,n,m-1} \tag{23}$$

and may be displayed as a pyramid made of triangular layers [22] as the Pascal trinomial coefficients do. The corresponding family of probability distributions is

$$p^{(\nu)}_{N,n,m} = \binom{N}{n, m} r^{(\nu)}_{N,n,m}. \tag{24}$$

Figure 5 shows probability distributions (24) for different values of ν and $N = 50$. For $\nu < 1$, ($\nu > 1$) a minimum (maximum) is observed at the barycenter of the triangle $0 \leqslant n + m \leqslant N$. For $\nu = 1$ the uniform distribution is obtained.

We shall now prove that family (24) yields a bidimensional Dirichlet distribution in the thermodynamic limit. With the change $t = 1 - \mathrm{e}^{-u}$ the first Beta function $B_1 \equiv B(N - n - m + \nu, n + m + 2\nu)$ in the numerator of (22) transforms as

$$B_1 = \int_0^1 t^{N-n-m+\nu-1}(1-t)^{n+m+2\nu-1}\mathrm{d}t \tag{25a}$$

$$= \int_0^\infty (1 - \mathrm{e}^{-u})^{N-n-m+\nu-1}\mathrm{e}^{-u(n+m+2\nu)}\mathrm{d}u \tag{25b}$$

$$= \int_0^\infty (1 - \mathrm{e}^{-u})^{\nu-1}\mathrm{e}^{-2\nu u}\mathrm{e}^{Nf(u)}\mathrm{d}u \tag{25c}$$

where now $f(u) = (1-x-y)\ln(1-\mathrm{e}^{-u}) - (x+y)u$, with $x = \frac{n}{N}$, $y = \frac{m}{N}$, $0 \leqslant x, y \leqslant 1$, $x+y \leqslant 1$, has a maximum at $u^\star = -\ln(x+y)$, with $f(u^\star) = (1-x-y)\ln(1-x-y) + (x+y)\ln(x+y)$

J. Stat. Mech. (2014) P12027

$\nu = 1/2$ $\nu = 1$

$\nu = 2$ $\nu = 5$

Figure 5. Probability distributions (24) for $\nu = 1/2$, 1, 2 and 5 and $N = 50$.

and $f''(u^\star) = -\frac{x+y}{1-x-y}$. Applying the Laplace method to integral (25c) yields

$$B_1 \simeq \sqrt{\frac{2\pi}{N}}\,(1-x-y)^{\nu-1}(x+y)^{2\nu} \times (1-x-y)^{N(1-x-y)+\frac{1}{2}}(x+y)^{N(x+y)-\frac{1}{2}}. \qquad (26)$$

Following the same steps in the second Beta function $B_2 \equiv B(n+\nu, m+\nu)$ in the numerator of (22) one gets

$$B_2 = \int_0^1 t^{n+\nu-1}(1-t)^{m+\nu-1}\mathrm{d}t \qquad (27a)$$

$$= \int_0^\infty (1-\mathrm{e}^{-u})^{n+\nu-1}\mathrm{e}^{-u(m+\nu)}\mathrm{d}u \qquad (27b)$$

$$= \int_0^\infty (1-\mathrm{e}^{-u})^{\nu-1}\mathrm{e}^{-\nu u}\mathrm{e}^{Nf(u)}\mathrm{d}u \qquad (27c)$$

where now function $f(u) = x\ln(1-\mathrm{e}^{-u}) - yu$ has a maximum at $u^\star = -\ln\left(\frac{y}{x+y}\right)$, with $f(u^\star) = x\ln x + y\ln y - (x+y)\ln(x+y)$, and $f''(u^\star) = -\frac{y}{x}(x+y)$. The Laplace procedure in integral (27c) yields

$$B_2 \simeq \sqrt{\frac{2\pi}{N}}\,x^{\nu-1}y^{\nu}(x+y)^{1-2\nu} \times x^{Nx+\frac{1}{2}}y^{Ny-\frac{1}{2}}(x+y)^{-N(x+y)-\frac{1}{2}}. \qquad (28)$$

J. Stat. Mech. (2014) P12027

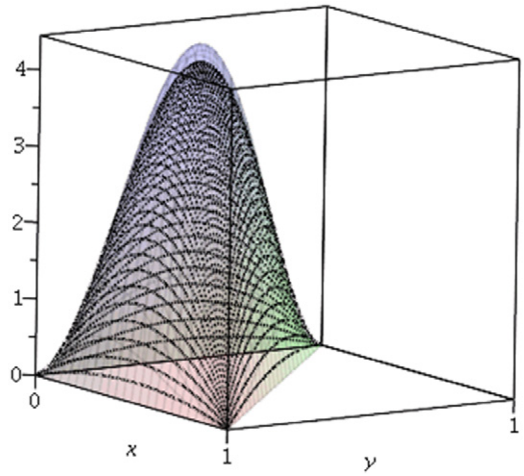

Figure 6. Bidimensional Dirichlet distribution (30) (solid surface) for $\nu = 2$ together with normalized distribution $N^2 p^{(2)}_{N,n,m}$ for $N = 50$ (dots).

Next, applying the Stirling approximation to the trinomial coefficients one gets

$$\binom{N}{n,m} \simeq \frac{1}{2\pi N} \frac{1}{x^{Nx+\frac{1}{2}} y^{Ny+\frac{1}{2}} (1-x-y)^{N(1-x-y)+\frac{1}{2}}}. \tag{29}$$

Finally, as the change $Nx = n$, $Ny = m$, transforms probability distribution (24) in $\mathcal{P}^{(\nu)}_N(x,y) = N^2 p^{(\nu)}_{N,n,m}$, taking into account (26), (28) and (29) yields

$$\lim_{N\to\infty} \mathcal{P}^{(\nu)}_N(x,y) = \mathcal{P}^{(\nu)}(x,y) = \frac{x^{\nu-1} y^{\nu-1} (1-x-y)^{\nu-1}}{B(\nu,\nu)B(\nu,2\nu)}; \quad x, y > 0,\, x + y < 1 \tag{30}$$

where $\frac{1}{B(\nu,\nu)B(\nu,2\nu)} = \frac{\Gamma(3\nu)}{\Gamma^3(\nu)}$, which is a bidimensional symmetric Dirichlet distribution with parameters $\alpha_1 = \alpha_2 = \alpha_3 = \nu$.

Figure 6 shows the limiting probability distribution (30) for $\nu = 2$ compared with the corresponding distribution $N^2 p^{(2)}_{N,n,m}$ versus $x = n/N$ and $y = m/N$ for $N = 50$. Larger values of N provide a closer aproach of both distributions. In order to get a one dimensional view of the convergence we shall resort to the marginal distributions. As a consequence of the property stated in section 4, the marginal distributions of symmetric Dirichlet distribution (30) are Beta distributions with parameters $\alpha_1 = \nu$, $\alpha_2 = 2\nu$. Figure 7 shows normalized marginal probability distributions $N\tilde{P}^{(\nu)}_{N,n}$ with $\tilde{P}^{(\nu)}_{N,n} \equiv \sum_{m=0}^{N-n} p^{(\nu)}_{N,n,m}$ for $\nu = 2$ and different values of N compared with the corresponding Beta distribution $f(x; 2, 4)$. The convergence is observed as expected.

5.3. $d \geqslant 3$

For $d = 3$, after some algebra, coefficients (12) can be expressed as

$$r^{(\nu)}_{N,n,m,l} = \frac{B(N-n-m-l-\nu, n+m+l+3\nu)}{B(\nu,\nu)} \times \frac{B(n+m+2\nu, l+\nu)B(n+\nu, m+\nu)}{B(\nu,2\nu)B(\nu,3\nu)} \tag{31}$$

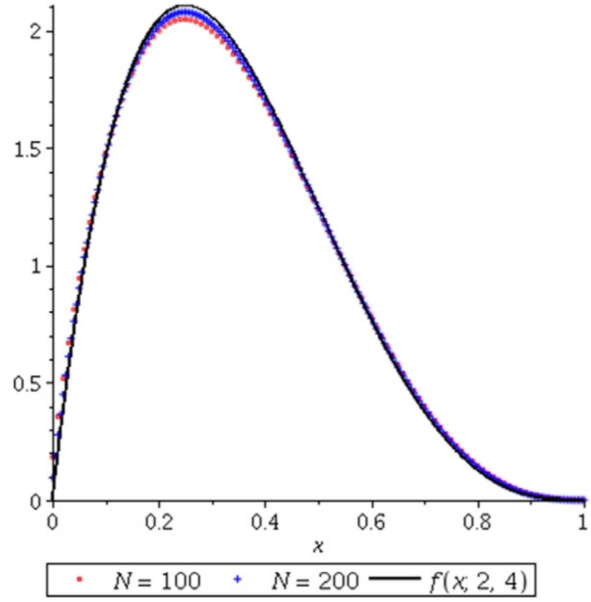

Figure 7. Marginal probability distributions $N\tilde{P}^{(2)}_{N,n}$ for $N = 100$ and 200 compared with the corresponding Beta distribution with parameters $\alpha_1 = 2$, $\alpha_2 = 4$.

with $n, m, l \geqslant 0$, $n + m + l \leqslant N$. Coefficients (31) obey the corresponding relation (14), which for $d = 3$ yields

$$r^{(\nu)}_{N,n,m,l} + r^{(\nu)}_{N,n+1,m-1,l} + r^{(\nu)}_{N,n,m-1,l+1} + r^{(\nu)}_{N,n,m-1,l} = r^{(\nu)}_{N-1,n,m-1,l} \tag{32}$$

and may be displayed as a hyperpyramid made of pyramids [22] as the Pascal tetranomial coefficients do. The corresponding family of probability distributions is

$$p^{(\nu)}_{N,n,m,l} = \binom{N}{n, m, l} r^{(\nu)}_{N,n,m,l}. \tag{33}$$

We shall not go through the details of the demonstration and simply show the corresponding approximations of the Beta functions $B_1 \equiv B(N - n - m - l - \nu, n + m + l + 3\nu)$, and $B_2 \equiv B(n + m + 2\nu, l + \nu)$ as

$$B_1 \simeq \sqrt{\frac{2\pi}{N}}\,(1 - x - y - z)^{\nu-1}(x + y + z)^{3\nu} \times (1 - x - y - z)^{N(1-x-y-z)+\frac{1}{2}}(x + y + z)^{N(x+y+z)-\frac{1}{2}} \tag{34a}$$

$$B_2 \simeq \sqrt{\frac{2\pi}{N}}\,(x + y)^{2\nu-1} z^{\nu}(x + y + z)^{1-3\nu} \times (x + y)^{N(x+y)+\frac{1}{2}} z^{Nz-\frac{1}{2}}(x + y + z)^{-N(x+y+z)-\frac{1}{2}}. \tag{34b}$$

On the other hand, the tetranomial coefficients may be approximated via the Stirling formula as

$$\binom{N}{n, m, l} \simeq \frac{1}{(2\pi N)^{\frac{3}{2}}} \frac{1}{x^{Nx+\frac{1}{2}} y^{Ny+\frac{1}{2}} z^{Nz+\frac{1}{2}} (1 - x - y - z)^{N(1-x-y-z)+\frac{1}{2}}}. \tag{35}$$

Finally, as the change $Nx = n$, $Ny = m$, $Nz = l$ transforms probability distribution (24) in $\mathcal{P}_N^{(\nu)}(x, y, z) = N^3 p_{N,n,m,l}^{(\nu)}$, taking into account (34*a*), (34*b*) and (35) yields

$$\lim_{N\to\infty} \mathcal{P}_N^{(\nu)}(x, y, z) = \mathcal{P}^{(\nu)}(x, y, z) = \frac{x^{\nu-1} y^{\nu-1} z^{\nu-1} (1 - x - y - z)^{\nu-1}}{B(\nu, \nu) B(\nu, 2\nu) B(\nu, 3\nu)}; \tag{36}$$

with $x, y, z, > 0$, $x + y + z < 1$, where $\frac{1}{B(\nu,\nu)B(\nu,2\nu)B(\nu,3\nu)} = \frac{\Gamma(4\nu)}{\Gamma^4(\nu)}$, which is a tridimensional symmetric Dirichlet distribution with parameters $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \nu$.

Following the above prescriptions it can be obtained the limiting probability distribution for any value of d as a d-dimensional symmetric Dirichlet distribution of parameter ν, due to the cancellation of intermediate terms in the product of Beta functions, though a general expression for the corresponding approximation of the involved Beta functions is too lengthy and tedious.

6. Conclusions

We have visited an up to now unexplored region of the q–β plane: that of the $\beta < 0$ half-plane, showing that a family of U-shaped q-Gaussians exist whenever $q > 2$ (for any dimension d), and can be interpreted as describing negative temperature systems. These distributions are in correspondence with symmetric Beta distributions with $\alpha_1 = \alpha_2 \in (0, 1)$ under the linear change (3).

We have analytically shown that discrete distributions (11), though having q-Gaussians as attractors in the $N \to \infty$ limit for $d = 1$, have instead Dirichlet limiting distributions for $d > 1$. Nevertheless, we should realize that, for $d > 1$, q-Gaussians have to do with symmetry. As mentioned in section 4, bidimensional Dirichlet distributions have symmetry of rotation of angle $2\pi/3$ whereas bidimensional q-Gaussians have cylindric symmetry. This is the reason why probability distributions (24) (symmetric under changes $n \leftrightarrow m \leftrightarrow N-n-m$) do not have bidimensional q-Gaussians as attractors. In this respect a possible generalization of probability distributions (24) based not in the triangular symmetry of trinomial coefficients (as well of coefficients (22)) but in some alternative generalized coefficients with square, pentagonal, hexagonal symmetry and so on (thus having symmetry of rotation of angle $2\pi/m$, with m being the number of sides of the polygon) would be the appropriate method to get q-Gaussian limiting distributions in the $m \to \infty$ limit. Work along this line certainly is welcome. Let us finally mention that for all the finite-N present sets of probabilities, the entropic functional which is extensive is the BG one, i.e. $S_{\mathrm{BG}}(N) \propto N$. This is due to the fact that, even if the N random variables are strongly correlated, the set of all admissible microscopic configurations is not heavily shrunken or modified.

Acknowledgments

We acknowledge fruitful remarks by G Ruiz, J P Gazeau and E M F Curado, as well as partial financial support from CNPq, Faperj and Capes (Brazilian agencies).

References

[1] Tsallis C 1988 *J. Stat. Phys.* **52** 479
[2] Tsallis C 2009 *Introduction to Nonextensive Statistical Mechanics* (New York: Springer)
[3] Prato D and Tsallis C 1999 *Phys. Rev.* E **60** 2398
[4] Umarov S and Tsallis C 2007 *AIP Conf. Proc.* **965** 34
[5] Umarov S, Tsallis C and Steinberg S 2008 *Milan J. Math.* **75** 307
[6] Plastino A R and Plastino A 1995 *Physica* A **222** 347
Tsallis C and Buckman D J 1996 *Phys. Rev.* E **54** R2197
[7] Combe G, Richefeu V, Viggiani G, Hall S A, Tengattini A and Atman A P F 2013 *AIP Conf. Proc.* **1542** 453
[8] Cirto L J L, de Assis V R V and Tsallis C 2014 *Physica* A **393** 286
Antoni M and Ruffo S 1995 *Phys. Rev.* E **52** 2361
[9] Richardson J D and Burlaga L F 2013 *Space Sci. Rev.* **176** 217
Burlaga L F and Vinas A F 2005 *Physica* A **356** 375
[10] Lutz E 2003 *Phys. Rev.* A **67** 051402
[11] Douglas P, Bergamini S and Renzoni F 2006 *Phys. Rev. Lett.* **96** 110601
[12] Lutz E and Renzoni F 2013 *Nat. Phys.* **9** 615
[13] Andrade J S, da Silva G F T, Moreira A A, Nobre F D and Curado E M F 2010 *Phys. Rev. Lett.* **105** 260601
[14] Upadhyaya A, Rieu J-P, Glazier J A and Sawada Y 2001 *Physica* A **293** 549
[15] Sire C and Chavanis P-H 2008 *Phys. Rev.* E **78** 061111
[16] Liu B and Goree J 2008 *Phys. Rev. Lett.* **100** 055003
[17] Livadiotis G and McComas D J 2014 *J. Plasma Phys.* **80** 341
[18] Yoon P H, Rhee T and Ryu C-M 2005 *Phys. Rev. Lett* **95** 215003
Yoon P H 2012 *Phys. Plasmas* **19** 012304
[19] Bains A S and Tribeche M 2014 *Astrohphys. Space Sci.* **351** 191
[20] DeVoe R G 2009 *Phys. Rev. Lett.* **102** 063001
[21] Tsallis C, Gell-Mann M and Sato Y 2005 *Proc. Natl Acad. Sci. USA* **102** 15377
[22] Rodríguez A and Tsallis C 2012 *J. Math. Phys.* **53** 023302
[23] Rodríguez A, Schwammle V and Tsallis C 2008 *J. Stat. Mech.* P09006
[24] Moyano L G, Tsallis C and Gell-Mann M 2006 *Europhys. Lett.* **73** 813
[25] Thistleton W J, Marsh J A, Nelson K P and Tsallis C 2009 *Cent. Eur. J. Phys.* **7** 3874
[26] Hilhorst H J 2010 *J. Stat. Mech.* P10023
[27] Hanel R, Thurner S and Tsallis C 2009 *Eur. Phys. J.* B **72** 263
[28] Bergeron H, Curado E M F, Gazeau J P and Rodrigues L M C S 2013 *J. Math. Phys.* **54** 123301
[29] Kotz S, Balakrishnan N and Johnson N L 2000 *Continuous Multivariate Distributions* (*Models and Applications* vol 1) (New York: Wiley)

J. Stat. Mech. (2014) P12027