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A generalization of the cumulant expansion. Application to a scale-invariant probabilistic model

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As well known, cumulant expansion is an alternative way to moment expansion to fully characterize probability distributions provided *all* the moments exist. If this is not the case, the so-called escort mean values (or q-moments) have been proposed to characterize probability densities with divergent moments [C. Tsallis *et al.*, J. Math. Phys. **50**, 043303 (2009)]. We introduce here a new mathematical object, namely, the *q-cumulants*, which, in analogy to the cumulants, provide an alternative characterization to that of the q-moments for the probability densities. To illustrate the technical details of the procedure, we apply this new scheme to further study a recently proposed family of scale-invariant discrete probabilistic models [A. Rodríguez *et al.*, J. Stat. Mech.: Theory Exp. **2008**, P09006; R. Hanel *et al.*, Eur. Phys. J. B **72**, 263 (2009)] having q-Gaussians as limiting probability distributions.
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I. INTRODUCTION

In classical thermodynamics for short-range-interacting systems we have two types of thermodynamical quantities, namely, the *extensive* (e.g., total energy, total entropy, volume, etc.) and the *intensive* (e.g., temperature, pressure, chemical potential, etc.) ones. Within the formalism of Boltzmann–Gibbs statistical mechanics, these quantities typically emerge as successive moments of the N-particle Hamiltonian. If, for whatever reason, exact calculations are not tractable, a variety of procedures exist, which involve truncations at some order. It is desirable that such truncations maintain the extensive or intensive nature of the quantities. Excepting the first moments in terms of the total Hamiltonian, which always are extensive, all the higher-order moments violate extensivity. There are, however, specific combinations of these moments that preserve the extensivity. These are the so-called cumulants. For example, the total specific heat, which has to be extensive, appears as a second-order cumulant, i.e., a convenient combination of first- and second-order moments. Naturally, cumulant expansions are mathematically legitimate *only when all moments are finite*. A variety of physical systems exist for which this property is not verified. The purpose of the present paper is to develop a generalized form of cumulant expansion that overcomes this restriction. These generalized cumulants are, in turn, based on consistently generalized moments (*escort moments* or *Q-moments*), which we shall introduce later on. In Sec. II we introduce the q-cumulants; in Sec. III we review a special family of scale-invariant probability models; in Sec. IV we study the corresponding Q-moments; and in Sec. V we address the associated (unnormalized) q-cumulants. We summarize our results in Sec. VI.

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Let us briefly recall the standard cumulant expansion. For a probability density function $f(x)$, the cumulants $\kappa_j = g^{(j)}(0)$, $j=1,2,\ldots$, are defined through the derivatives of the cumulant-generating function

$$g(t) \equiv \ln M(t) = t\kappa_1 + \frac{t^2}{2!}\kappa_2 + \frac{t^3}{3!}\kappa_3 + \cdots, \tag{1}$$

where

$$M(t) \equiv \langle e^{tx} \rangle = 1 + t\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \cdots \tag{2}$$

is the moment-generating function and $\mu_j = \langle x^j \rangle = M^{(j)}(0)$, $j=0,1,2,\ldots$, are the moments. Taking logarithms and Taylor expanding in Eq. (2) yields

$$\begin{aligned}
\ln M(t) = {} & \frac{t}{1!}\mu_1 \\
& + \frac{t^2}{2!}(\mu_2 - \mu_1^2) \\
& + \frac{t^3}{3!}(\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3) \\
& + \frac{t^4}{4!}(\mu_4 - 4\mu_1\mu_3 + 12\mu_1^2\mu_2 - 3\mu_2^2 - 6\mu_1^4) \\
& + \frac{t^5}{5!}(\mu_5 - 5\mu_1\mu_4 - 10\mu_2\mu_3 + 20\mu_1^2\mu_3 - 60\mu_1^3\mu_2 + 30\mu_1\mu_2^2 + 24\mu_1^5) \\
& + \frac{t^6}{6!}(\mu_6 - 10\mu_3^2 - 15\mu_2\mu_4 + 30\mu_2^3 - 6\mu_1\mu_5 + 120\mu_1\mu_2\mu_3 + 30\mu_1^2\mu_4 \\
& - 270\mu_1^2\mu_2^2 - 120\mu_1^3\mu_3 + 360\mu_2\mu_4 - 120\mu_6) \\
& \vdots
\end{aligned} \tag{3}$$

Comparing Eqs. (3) and (1), one gets $\kappa_1 = \mu_1$, $\kappa_2 = \mu_2 - \mu_1^2$, $\kappa_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3, \ldots$, which follow the general relation between moments and cumulants given by

$$\kappa_j = j! \sum_{\Sigma_{i=1}^{j} i n_i = j} \frac{(-1)^{\bar{n}-1}}{\bar{n}} \binom{\bar{n}}{n_1, n_2, \cdots, n_j} \prod_{i=1}^{j} \left(\frac{\mu_i}{i!}\right)^{n_i}, \quad j = 1,2,\ldots, \tag{4}$$

where the sum runs over solutions of the equation $\Sigma_{i=1}^{j} i n_i = j$, with n_i, $i=1,2,\ldots,j$ being nonnegative integers, $\bar{n} = \Sigma_{i=1}^{j} n_i$ and we have made use of the multinomial coefficient $\binom{\bar{n}}{n_1, n_2 \cdots, n_j} = \bar{n}!/(n_1! n_2! \cdots n_j!)$. Relation (4) is a convenient rewriting of the relations given in Refs. 1 and 2. An alternative connection between cumulants and moments can be seen in Ref. 3.

As well known, the set of moments fully characterize a probability density function provided they are *all finite*, the set of cumulants (linear combinations of the moments) being an alternative and, for some purposes (see later on), more convenient description. Once the set of moments are known, the probability distribution may be obtained via Fourier antitransforming the characteristic function $\varphi(t) = M(it) = \mathcal{F}[f](t) = \int_{-\infty}^{\infty} dx e^{itx} f(x)$.

Notice that although $\kappa_2 = \langle (x - \langle x \rangle)^2 \rangle$ and $\kappa_3 = \langle (x - \langle x \rangle)^3 \rangle$, in general, the cumulants do *not* coincide with the centered moments since $\kappa_4 \neq \langle (x - \langle x \rangle)^4 \rangle$ and so on.

Let us review now an important property. We assume that $x = \xi_1 + \xi_2 + \cdots + \xi_N$, where $\{\xi_l\}$ are *any N equal* and *independent* random variables. We straightforwardly verify that

$$\langle x\rangle = N\langle \xi_1\rangle,$$

$$\langle x^2\rangle - \langle x\rangle^2 = N[\langle \xi_1^2\rangle - \langle \xi_1\rangle^2],$$

$$\langle x^3\rangle - 3\langle x^2\rangle\langle x\rangle + 2\langle x\rangle^3 = N[\langle \xi_1^3\rangle - 3\langle \xi_1^2\rangle\langle \xi_1\rangle + 2\langle \xi_1\rangle^3]$$

$$\vdots \tag{5}$$

In general,

$$\kappa_j(N) = N\kappa_j(1) \quad (j = 1,2,3,\ldots;\ N = 1,2,3,\ldots), \tag{6}$$

where the notation $\kappa_j(N)$ is self-explanatory. In other words, *all* cumulants are *extensive* in the thermodynamical sense. The main purpose of the present paper is to discuss what happens with this property in the presence of strong correlations such as those that are typical within q-statistics,[4] a current generalization of Boltzmann–Gibbs statistical mechanics (recovered as the $q=1$ particular instance). This generalization, sometimes referred to as *nonextensive statistical mechanics*, has received a wide variety of physical applications.[5–19]

II. *q*-CUMULANTS

Within the frame of q-statistics, a generalization of the concept of cumulant is in order. A first natural attempt for such generalization consists of just replacing the logarithm and exponential functions in (3) by the q-logarithm and q-exponential functions (see Appendix for definitions and corresponding expansions). Their respective Taylor expansions yields

$$\begin{aligned}
\ln_q\langle e_q^{tx}\rangle &= t\kappa_{1,q} + \frac{t^2}{2!}\kappa_{2,q} + \frac{t^3}{3!}\kappa_{3,q} + \frac{t^4}{4!}\kappa_{4,q} + \frac{t^5}{5!}\kappa_{5,q} + \cdots \\
&= \frac{t}{1!}\mu_1 \\
&\quad + \frac{t^2}{2!}q(\mu_2 - \mu_1^2) \\
&\quad + \frac{t^3}{3!}q((2q-1)\mu_3 - 3q\mu_1\mu_2 + (q+1)\mu_1^3) \\
&\quad + \frac{t^4}{4!}q((2q-1)(3q-2)\mu_4 - 4q(2q-1)\mu_1\mu_3 + 6q(q+1)\mu_1^2\mu_2 - 3q^2\mu_2^2 - (q+1)(q \\
&\quad + 2)\mu_1^4) \\
&\quad + \frac{t^5}{5!}q((2q-1)(3q-2)(4q-3)\mu_5 - 5q(2q-1)(3q-2)\mu_1\mu_4 - 10q^2(2q-1)\mu_2\mu_3 \\
&\quad + 10(q+1)q(2q-1)\mu_1^2\mu_3 - 10q(q+1)(q+2)\mu_1^3\mu_2 + 15q^2(q+1)\mu_1\mu_2^2 \\
&\quad + (q+1)(q+2)(q+3)\mu_1^5) \\
&\quad \vdots
\end{aligned} \tag{7}$$

where we have defined the *q-cumulant of order j,* $\kappa_{j,q}$, which is related to the moments through

$$\kappa_{j,q} = j! \sum_{\Sigma_{i=1}^{j} i n_i = j} \frac{(-1)^{\bar{n}-1}}{\bar{n}!} l(q,\bar{n}) \binom{\bar{n}}{n_1, n_2, \cdots, n_j} \prod_{i=1}^{j} \left(\frac{e(q,i)\mu_i}{i!} \right)^{n_i}, \quad j = 1,2, \ldots, \tag{8}$$

where $e(q,n)$ and $l(q,n)$ are the coefficients of the Taylor expansion of the q-exponential and q-logarithm functions respectively (see the Appendix).

Notice that the moments $\{\mu_j\}$ in (7) are exactly the same as those introduced in Eqs. (2) and (3). In other words, $\{\mu_j\}$ does *not* depend on q. Notice also that $\kappa_{1,q} = \mu_1$, $\forall q$ (the q-cumulant of first order coincide with the first moment for any value of q) and $\kappa_{j,1} = \kappa_j$, $\forall j$ (the q-cumulants reduce to the standard cumulants for $q=1$).

Although interesting, in principle, we will show that the q-generalization, which is useful, is *not* the above one, but the one which follows now. We will define the *unnormalized* q-cumulants, $\kappa'_{j,q}$, through

$$\begin{aligned}
\ln_q M_q(t) &= t\kappa'_{1,q} + \frac{t^2}{2!}\kappa'_{2,q} + \frac{t^3}{3!}\kappa'_{3,q} + \frac{t^4}{4!}\kappa'_{4,q} + \frac{t^5}{5!}\kappa'_{5,q} + \cdots \\
&= \frac{t}{1!}\mu'_{1,q} \\
&\quad + \frac{t^2}{2!} q(\mu'_{2,2q-1} - \mu'^{2}_{1,q}) \\
&\quad + \frac{t^3}{3!} q((2q-1)\mu'_{3,3q-2} - 3q\mu'_{2,2q-1}\mu'_{1,q} + (q+1)\mu'^{3}_{1,q}) \\
&\quad + \frac{t^4}{4!} q((2q-1)(3q-2)\mu'_{4,4q-3} - 4q(2q-1)\mu'_{3,3q-2}\mu'_{1,q} + 6q(q+1)\mu'_{2,2q-1}\mu'^{2}_{1,q} \\
&\quad - 3q^2\mu'^{2}_{2,2q-1} - (q+1)(q+2)\mu'^{4}_{1,q}) \\
&\quad + \frac{t^5}{5!} q((2q-1)(3q-2)(4q-3)\mu'_{5,5q-4} - 5q(2q-1)(3q-2)\mu'_{4,4q-3}\mu'_{1,q} \\
&\quad - 10q^2(2q-1)\mu'_{3,3q-2}\mu'_{2,2q-1} + 10(q+1)q(2q-1)\mu'_{3,3q-2}\mu'^{2}_{1,q} \\
&\quad - 10q(q+1)(q+2)\mu'_{2,2q-1}\mu'^{3}_{1,q} + 15q^2(q+1)\mu'^{2}_{2,2q-1}\mu'_{1,q} \\
&\quad + (q+1)(q+2)(q+3)\mu'^{5}_{1,q}) \\
&\quad \vdots
\end{aligned} \tag{9}$$

where we shall call $M_q(t) \equiv \langle e_q^{txf(x)^{q-1}} \rangle$ the *Q-moments generating function*. Expression (9) makes use of the so-called *unnormalized escort moments*,[20] $\mu'_{j,Q} \equiv \int dx x^j f(x)^Q$, related, in turn, to the standard *escort moments* (or *Q-moments*), $\mu_{j,Q} = \mu'_{j,Q} / \int dx f(x)^Q$, i.e., x^j averaged over the *escort distribution* $f_Q(x) \equiv f(x)^Q / \gamma_Q$, with $\gamma_Q \equiv \int dx f(x)^Q$. In the limit $Q \to 1$, the escort distribution as well as the Q-moments, and the unnormalized Q-moments, tend to the standard ones.

Notice from Eq. (9) that for the first unnormalized q-cumulant we must use the Q-moment with $Q=q$ (in fact, the unnormalized q-cumulant coincides with the unnormalized escort moment for $j=1: \kappa'_{1,q} = \mu'_{1,q}$), for the second unnormalized q-cumulant, Q-moments with two different values, namely, $Q=q$ and $Q=2q-1$, must be used, and similarly for higher-order unnormalized q-cumulants. The relation between the unnormalized q-cumulants and the unnormalized Q-moments is thus given by

$$\kappa'_{j,q} = j! \sum_{\sum_{i=1}^{j} i n_i = j} \frac{(-1)^{\bar{n}-1}}{\bar{n}!} l(q,\bar{n}) \binom{\bar{n}}{n_1, n_2, \cdots, n_j} \prod_{i=1}^{j} \left(\frac{e(q,i)\mu'_{i,i(q-1)+1}}{i!} \right)^{n_i}. \tag{10}$$

For $q=1$, the unnormalized escort moments appearing in Eqs. (9) and (10): $\mu'_{i,Q}$, with $Q=i(q-1)+1$, $i=1,2,3,\ldots$, reduce to the normalized escort moments, which, in turn, coincide with the ordinary moments: $\mu'_{i,1}=\mu_{i,1}=\mu_i$, $\forall i$. The same happens with the unnormalized q-cumulants: $\kappa'_{i,1}=\kappa_{i,1}=\kappa_i$, $\forall i$, so Eqs. (4), (8), and (10) coincide.

For the general $q \neq 1$ case, as shown in Ref. 20, the family of unnormalized escort moments $\mu'_{i,i(q-1)+1}$, $i=1,2,3,\ldots$, provided they are *all finite*, may serve, together with their accompanying denominators, $\gamma_{i(q-1)+1}=\int dx f(x)^{i(q-1)+1}$, in the expression of the corresponding escort moments, as an alternative description for the quite frequent cases of probability density functions whose moments are not defined [e.g., moments on the order of $j \geq 2$ for *q-Gaussians* with $q \geq 5/3$ (Ref. 21)]. In analogy with the $Q=1$ case, once the Q-moments generating function is known, the probability distribution may be obtained via q-Fourier antitransforming the *q-characteristic function* $\varphi_q(t)=M_q(it) \equiv \mathcal{F}_q[f](t)=\int_{-\infty}^{\infty} dx e_q^{itxf(x)^{q-1}} f(x)$ $(f(x) \geq 0,\ q \geq 1)$, where we have made use of the so-called *q-Fourier transform*,[22,23] which is a (nonlinear) q-generalization of the standard Fourier transform.

III. SCALE INVARIANT TRIANGLES

We will apply the above formalism to further study a family of scale-invariant probabilistic models characterized by a real number $\nu > 0$, first introduced in Ref. 24 and later generalized in Ref. 25. The model consists of a set of N *equal*, *long-range-correlated binary* random variables $\{x_i\}$, which take values of 0 and 1.

For the 2^N elementary events of the sample space of the N variable system, there is a set of only $(N+1)$ different probabilities given by[25]

$$r_{N,n}^{(\nu)} = \frac{B(n+\nu, N-n+\nu)}{B(\nu,\nu)} \tag{11}$$

for $n=0,1,\ldots,N$, where $B(x,y)$ stands for the beta function. The $\{r_{N,n}^{(\nu)}\}$ satisfies the so-called Leibnitz triangle rule[24]

$$r_{N,n}^{(\nu)} + r_{N,n+1}^{(\nu)} = r_{N-1,n}^{(\nu)}, \tag{12}$$

which is a fingerprint of the scale-invariant character of the system, which states that the marginal probability distribution of the N system coincides with the joint probability distribution of the corresponding $(N-1)$ subsystem. In other words, in its continuous version, we have $\int p_N(x_1,x_2,\ldots,x_N)dx_N = p_{N-1}(x_1,x_2,\ldots,x_{N-1})$. [Although similar in form, these conditions are not the Kolmogorov consistency conditions of a stochastic process (see, for instance, Ref. 26). Indeed, the Kolmogorov conditions refer to a system of N elements, whereas here we impose nontrivial conditions connecting the joint probabilities of a $(N-1)$-subsystem with the marginal ones of a N-system.]

For a fixed value of ν, the probability (11) may be displayed as a function of N in a symmetric (i.e., $r_{N,n}^{(\nu)}=r_{N,N-n}^{(\nu)}$) triangle (thus, the name *scale-invariant triangles*). As an example, the $\nu=\frac{1}{2}$ triangle reads as

TABLE I. The two different changes of variables corresponding, respectively, to all real values $q_{\text{lim}}<1$ (Refs. 24 and 25) and all real values $1<q_{\text{lim}}<3$ (Ref. 25) and the associated corresponding $\mathcal{P}^{(\nu)}(x)$ and q_{lim}.

Map	$\mathcal{P}^{(\nu)}(y)$	q_{lim}
$y=2\sqrt{\nu-1}\left(\frac{n}{N}-\frac{1}{2}\right)$	$\frac{N}{2\sqrt{\nu-1}}p_{N,n}^{(\nu)}$	$q_\nu=\frac{\nu-2}{\nu-1}<1;\quad(\nu>1)$
$y=\sqrt{\nu+\frac{1}{2}}\frac{\left(n-\frac{N}{2}\right)}{\sqrt{n(N-n)}}$	$\frac{4(n(N-n))^{3/2}}{N^2\sqrt{\nu+\frac{1}{2}}}p_{N,n}^{(\nu)}$	$\bar{q}_\nu=\frac{2\nu+3}{2\nu+1}>1;\quad(\nu>0)$

$$
\begin{array}{lccccccccccc}
 & & & & & r_{N,n}^{(1/2)} & & & & & & \\
(N=0) & & & & & 1 & & & & & & \\
(N=1) & & & & \frac{1}{2} & & \frac{1}{2} & & & & & \\
(N=2) & & & \frac{3}{8} & & \frac{1}{8} & & \frac{3}{8} & & & & \\
(N=3) & & \frac{5}{16} & & \frac{1}{16} & & \frac{1}{16} & & \frac{5}{16} & & & \\
(N=4) & \frac{35}{128} & & \frac{5}{128} & & \frac{3}{128} & & \frac{5}{128} & & \frac{35}{128} & & \\
(N=5) & \frac{63}{256} & & \frac{7}{256} & & \frac{3}{256} & & \frac{7}{256} & & \frac{1}{256} & & \frac{63}{256} \\
 & \vdots & & & & \vdots & & & & \vdots & &
\end{array}
$$

Due to the properties of the beta function, the coefficients of the triangle $r_{N,n}^{(\nu)}$ are rational numbers for any rational value of ν.

Special values of ν are $\nu=1$, for which the original Leibnitz triangle[27] is obtained $\left[1/r_{N,n}^{(1)}=(N+1)\binom{N}{n}\right]$, and the limiting case $r_{N,n}^{(\infty)}\equiv\lim_{\nu\to\infty}r_{N,n}^{(\nu)}=1/2^N$, for which the system becomes uncorrelated.[24] In the following, we shall refer to these limiting cases as the Leibnitz and the de Moivre–Laplace (or Boltzmann–Gibbs) limits, respectively.

The probability distribution for the discrete variable $z=x_1+x_2+\cdots+x_N$, which takes values of $0,1,\ldots,N$, is given by

$$p_{N,n}^{(\nu)}\equiv P(z=N-n)=\binom{N}{n}r_{N,n}^{(\nu)} \tag{13}$$

with $\Sigma_{n=0}^{N}p_{N,n}^{(\nu)}=1$, due to the degeneracy imposed by the identical character of the N binary variables. By properly centering and scaling with a map $n\to y(n)$, the distribution (13) is transformed into a new one, $\mathcal{P}^{(\nu)}(y)$, which tends to a q-Gaussian

$$\mathcal{P}^{(\nu)}(y)\to A_q e_q^{-y^2} \tag{14}$$

for $N\to\infty$, with A_q such $\int A_q e_q^{-y^2}dy=1$, and a value of $q=q_{\text{lim}}$, which depends on ν and the chosen change of variable. The variable change, the intermediate distribution, and the value of q_{lim} are given in Table I. The Leibnitz case ($\nu=1$, hence $q\to-\infty$) corresponds to a uniform distribution, while for the Boltzmann–Gibbs case ($\nu=\infty$, hence $q=1$), the standard Gaussian is obtained.

Thus, depending on the chosen discretization (namely, Table I, top or bottom discretization), for each value of $\nu\geq1$, there exist two conjugated q-Gaussians: one of them with $1>q=q_\nu\in(-\infty,1)$ (with compact support $|x|<1/\sqrt{1-q}$) and the other with $1<q=\bar{q}_\nu\in(1,5/3]$ (the support being the whole real axis). In the case of ν being a positive integer, the different values of q are shown in Table II. For $\nu\in(0,1)$, however, only one q-Gaussian with $1<q=\bar{q}_\nu\in(5/3,3)$ is obtained.[25] Figure 1 plots the two different q_{lim} values as a function of ν.

Relations $q(\nu)$ given in the third column of Table I can be inverted so as to express the family of triangles as a function of q,[25]

TABLE II. Values of q as a function of ν for the two families of conjugated q-Gaussians.

ν	0^+	1/2	1	2	3	4	$\cdots$	∞
q_ν			$-\infty$	0	1/2	2/3	$\cdots$	1^-
$\bar{q}_\nu$	3^-	2	5/3	7/5	9/7	11/9	$\cdots$	1^+

$$r_{N,n}^{(q)} = \begin{cases} \dfrac{B\left(n+\dfrac{q-2}{q-1}, N-n+\dfrac{q-2}{q-1}\right)}{B\left(\dfrac{q-2}{q-1}, \dfrac{q-2}{q-1}\right)}, & q<1 \\ \dfrac{1}{2^N}, & q=1 \\ \dfrac{B\left(n+\dfrac{3-q}{2(q-1)}, N-n+\dfrac{3-q}{2(q-1)}\right)}{B\left(\dfrac{3-q}{2(q-1)}, \dfrac{3-q}{2(q-1)}\right)}, & 1<q<3, \end{cases} \tag{15}$$

which allows us to fix *a priori* the value of q and then get the triangle generating the corresponding q-Gaussian.

Figure 2 shows the probability distribution $\mathcal{P}^{(\nu)}(x)$ for an $N=1000$ system and two different values of the parameter ν, namely, $\nu=2$, for which $q_\nu=0<1$ and $\bar{q}_\nu=7/5>1$, and $\nu=1/2$, for which there exist one only value $\bar{q}_\nu=2$. The corresponding q-Gaussians (not shown) graphically overlap with the shown distributions. To further check Eq. (14), Fig. 3 shows $\log_{q_\nu}(\mathcal{P}^{(\nu)}(x)/\mathcal{P}^{(\nu)}(0))$ versus $(x\mathcal{P}^{(\nu)}(0))^2$—which, in the $N\to\infty$ limit, coincides with a straight line for a q-Gaussian—for $\nu=1/2$, $\bar{q}_\nu=7/5$ (left) and $\nu=2$, $\bar{q}_\nu=2$ (right) for $N=200$, 500, and 1000. As can be seen in the figure, points far from the origin (corresponding to small values $n=0,1,\ldots$, and their symmetric $n=N-1,N-2,\ldots$) deviate from the fitting to a straight line. Nevertheless, this is a finite size effect since increasing N has the effect of bringing them back to the line. For values of ν and discretizations corresponding to compact support q-Gaussians (not shown), this finite size effect is not present and the fitting is improved.

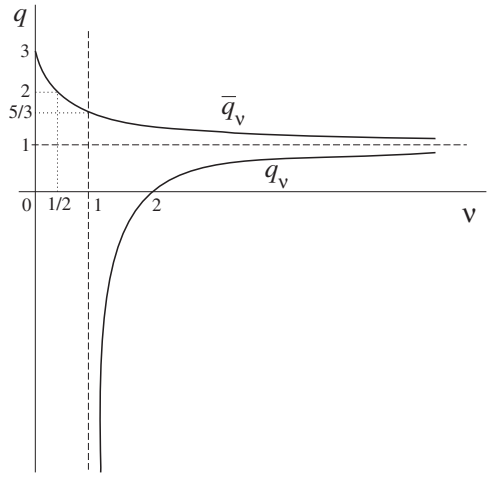

FIG. 1. Limiting values q_ν and $\bar{q}_\nu$ as a function of ν. Notice that the $\nu(q)$ is a single-valued function, whereas $q(\nu)$ is not.

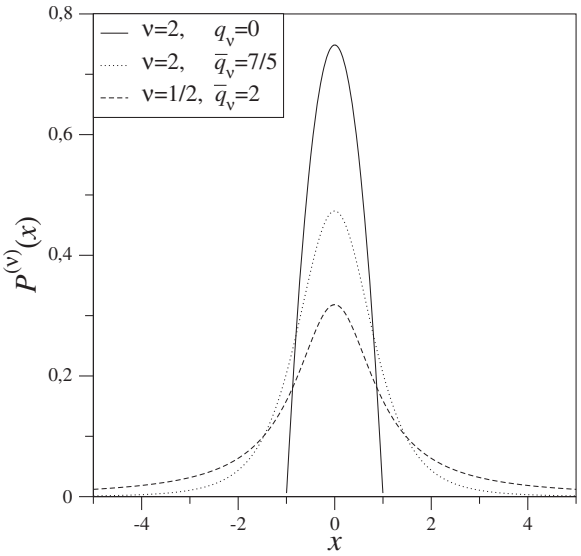

FIG. 2. Probability distribution $\mathcal{P}^{(\nu)}(x)$ for $\nu=2$, $q_\nu=0$ (solid line), $\nu=2$, $\bar{q}_\nu=7/5$ (dotted line), and $\nu=1/2$, $\bar{q}_\nu=2$ (dashed line). $N=1000$ in all cases.

IV. SCALING OF THE *Q*-MOMENTS

Our aim in this section is to study the scaling with the system size of the Q-moments and q-cumulants of the family of triangles (11) or, equivalently, (15).

For our model, the escort moments of interest read as

$$\mu_{j,Q}^{(\nu)} \equiv \langle y^j \rangle_Q = \frac{\Sigma_{n=0}^N n^j (p_{N,n}^{(\nu)})^Q}{\Sigma_{n=0}^N (p_{N,n}^{(\nu)})^Q} = \frac{\mu'^{(\nu)}_{j,Q}}{\gamma_Q^{(\nu)}}, \tag{16}$$

where the numerator of the last fraction stands for the corresponding unnormalized Q-moment, the denominator being the normalization factor of the escort distribution.

Straightforward calculations based on the symmetric character of triangles (11) allows us to exactly calculate the first-order escort moments as

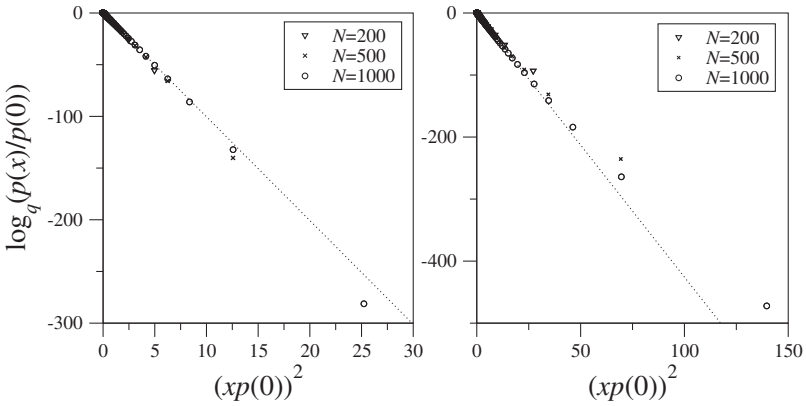

FIG. 3. $\log_{q_\nu}(\mathcal{P}^{(\nu)}(x)/\mathcal{P}^{(\nu)}(0))$ vs $(x\mathcal{P}^{(\nu)}(0))^2$ for $\nu=1/2$, $\bar{q}_\nu=7/5$ (left) and $\nu=2$, $\bar{q}_\nu=2$ (right) for $N=200$ (triangles), 500 (crosses), and 1000 (circles). The dotted straight line is obtained by linear fitting after dropping the last four points.

$$\mu_{1,Q}^{(\nu)} = \kappa_{1,Q}^{(\nu)} = \frac{N}{2}, \quad \forall\, Q, \quad \forall\, \nu. \tag{17}$$

The higher-order escort moments for the aforementioned Boltzmann–Gibbs and Leibnitz limiting cases are addressed next.

A. Boltzmann–Gibbs limit

To begin with, we shall study the standard $(Q=1)$ moments for the Boltzmann–Gibbs $(\nu \to \infty)$ limit. In this case the probability distribution (13) reduces to the binomial distribution $p_{N,n}^{(\infty)} = (1/2^N)\binom{N}{n}$ (which tends to the Gaussian distribution in the thermodynamic limit $N \to \infty$). Taylor expanding its moment-generating function $M(t) = \langle e^{tx} \rangle = (1/2^N)\Sigma_{n=0}^{N}\binom{N}{n}e^{tn} = ((1+e^t)/2)^N$, the standard moments $\mu_{j,1}^{(\infty)} = M^{(j)}(0)$ of the binomial distribution are easily obtained,

$$\mu_{1,1}^{(\infty)} = \frac{N}{2},$$

$$\mu_{2,1}^{(\infty)} = \frac{N(N+1)}{4},$$

$$\mu_{3,1}^{(\infty)} = \frac{N^2(N+3)}{8},$$

$$\mu_{4,1}^{(\infty)} = \frac{N(N^3+6N^2+3N-2)}{16},$$

$$\mu_{5,1}^{(\infty)} = \frac{N(N^4+10N^3+15N^2-10N)}{32},$$

$$\vdots \tag{18}$$

Though not shown here, it is worthy noticing the unexpected result that the scaling law (18) holds for the escort moments of the binomial distribution for *any value of* Q, that is, for the binomial distribution, one gets

$$\mu_{j,Q}^{(\infty)} \sim \frac{1}{2^j} N^j, \quad \forall\, Q. \tag{19}$$

On the other hand, the standard $(q=1)$ cumulants $\kappa_{j,1}^{(\infty)}$ of the binomial distribution can be obtained making use of Eq. (4) or, alternatively, as the derivatives of the cumulant-generating function $g(t) = N \ln((1+e^t)/2)$,

$$\kappa_{1,1}^{(\infty)} = \frac{N}{2}, \quad \kappa_{2,1}^{(\infty)} = \frac{N}{4}, \quad \kappa_{3,1}^{(\infty)} = 0, \quad \kappa_{4,1}^{(\infty)} = -\frac{N}{8}, \quad \kappa_{5,1}^{(\infty)} = 0, \quad \ldots, \tag{20}$$

where all the odd cumulants vanish from the third on. It has to be noticed that for $N \to \infty$, while the moments follow the rule $\mu_{j,1}^{(\infty)} \propto N^j$, the cumulants remain extensive, that is, $\kappa_{j,1}^{(\infty)} \propto N$. In fact, we easily verify that the results in Eq. (20) satisfy Eq. (6).

B. Leibnitz limit

In the specially simple Leibnitz case $(\nu=1)$, the probability distribution (13) reduces to the uniform one, $p_{N,n}^{(1)} = 1/(N+1)$, which once introduced in Eq. (16) yields $\mu_{j,Q}^{(1)} = [1/(N+1)]\Sigma_{n=0}^{N} n^j$ *for*

all Q, so the escort moments are independent of the value of Q, thus coinciding with the standard $(Q=1)$ moments, which may again be calculated as the derivatives of the corresponding moment-generating function $M(t)=\langle e^{tx}\rangle=[1/(N+1)]\Sigma_{n=0}^{N}e^{tn}=[1/(N+1)][(1-e^{(N+1)t})/(1-e^{t})]$, $M(0)=1$,

$$\mu_{1,Q}^{(1)}=\frac{N}{2},$$

$$\mu_{2,Q}^{(1)}=\frac{N^2}{3}+\frac{N}{6},$$

$$\mu_{3,Q}^{(1)}=\frac{N^3}{4}+\frac{N^2}{4},$$

$$\mu_{4,Q}^{(1)}=\frac{N^4}{5}+3\frac{N^3}{10}+\frac{N^2}{30}-\frac{N}{30},$$

$$\mu_{5,Q}^{(1)}=\frac{N^5}{6}+\frac{N^4}{3}+\frac{N^3}{12}-\frac{N^2}{12},$$

$$\vdots \tag{21}$$

In turn, the standard $(q=1)$ cumulants $\kappa_{j,1}^{(1)}$ of the discrete uniform distribution can be obtained through Eq. (4) or from the cumulant-generating function $g(t)=\ln([1/(N+1)]\times[(1-e^{(N+1)t})/(1-e^{t})])$, $g(0)=0$,

$$\kappa_{1,1}^{(1)}=\frac{N}{2},$$

$$\kappa_{2,1}^{(1)}=\frac{N^2}{12}+\frac{N}{6},$$

$$\kappa_{3,1}^{(1)}=0,$$

$$\kappa_{4,1}^{(1)}=-\frac{N^4}{120}-\frac{N^3}{30}-\frac{N^2}{20}-\frac{N}{30},$$

$$\kappa_{5,1}^{(1)}=0,$$

$$\vdots \tag{22}$$

where again the odd cumulants of order greater than 1 vanish. Scaling of cumulants (22) may be expressed as $\kappa_{j,1}^{(1)}\sim(B_j/j)N^j$, with B_j being the Bernoulli numbers $(B_1=1/2,\ B_2=1/6,\ B_3=0,\ B_4=-1/30,\ B_5=0,\ B_6=1/42,\ldots)$.

C. General case

We shall now allow ν to take on any positive value in (16). In principle, Q may take on any real value. However, as we will be interested in the escort moments involved in the expression of the unnormalized q-cumulants (10), we will first focus on the case $Q=j'(q-1)+1$ [the extra index

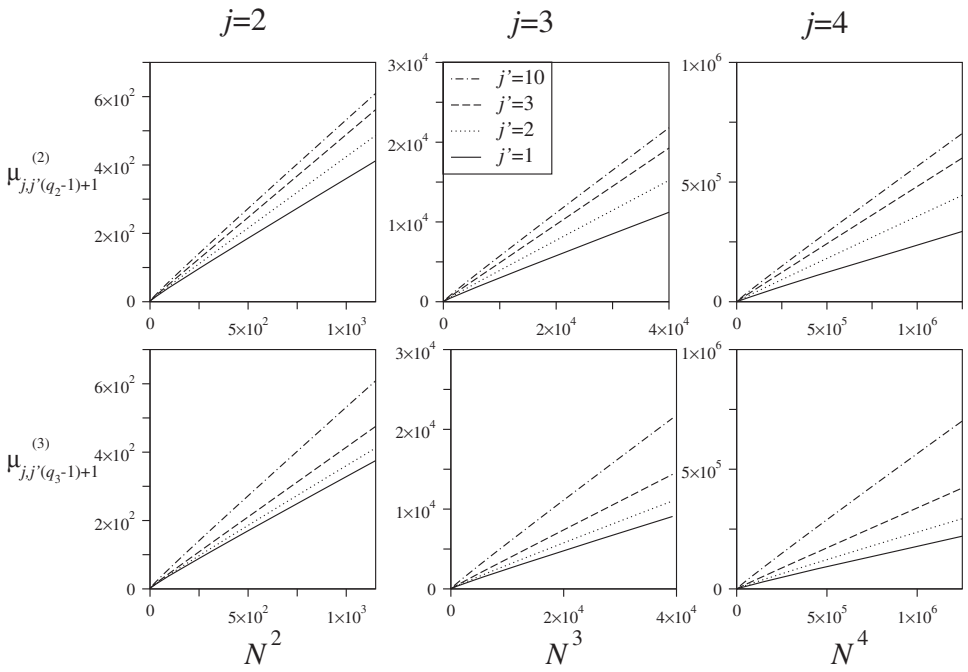

FIG. 4. Scaling of the escort moments (16) with $Q=j'(q_\nu-1)+1$ for different values of j': for $\nu=2$ (up), $\nu=3$ (down) and $j=2$ (left), $j=3$ (center), and $j=4$ (right). The behavior for $q=\bar{q}_\nu$ is totally analogous. For the not shown $j=1$ case, all the curves collapse in one: $\mu^{(\nu)}_{1,j'(q_\nu-1)+1}=N/2$ since relation (17) is satisfied.

j' has been introduced in order to allow for a more general description, though the only escort moments appearing in Eq. (10) are those with $j'=j$]. Furthermore, for the escort moment of the ν triangle we will take either $q=q_\nu$ or $q=\bar{q}_\nu$ as given in Table I.

Figure 4 shows the scaling of the escort moments $\mu^{(\nu)}_{j,j'(q_\nu-1)+1}$ on the orders of $j=2$, 3, and 4, for $\nu=2$ and 3 and different values of j'. The following trend is observed:

$$\mu^{(\nu)}_{j,j'(q_\nu-1)+1} \sim A_{j,j',q_\nu} N^j, \quad \forall\, \nu, \quad \forall\, j', \tag{23}$$

so the exponent of the scaling of the escort moments depends *only* on the order of j of the moment not on j', neither on q_ν, which considerably simplifies the discussion. From now on, we will restrict ourselves to the case $j'=j$ [and correspondingly denote $A_{j,q_\nu}\equiv A_{j,j'=j,q_\nu}$ in (23)]. Not shown results reveal that Eq. (23) is also valid for $Q=j'(q-1)+1$ with $q=\bar{q}_\nu$.

The prefactor A_{j,q_ν} in the scaling law (23) has already been analytically obtained in the Leibnitz ($q=-\infty$) case as $A_{j,-\infty}=\frac{1}{(j+1)}$ as well as in the opposite limit, the Boltzmann–Gibbs ($q=1$) case as $A_{j,1}=1/2^j$ [see Eqs. (21) and (18), respectively). A transition between both behaviors is expected for the limit $q\to 1^{\pm}$. In order to numerically obtain A_{j,q_ν}, Fig. 5(a) shows the ratio $\mu^{(\nu)}_{j,j(q_\nu-1)+1}/N^j$ for $\nu=50$ as a function of N for different values of j. It is observed that $A_{j,q_{50}} \sim 1/2^j$. Figure 5(b) shows again A_{j,q_ν}, now approximated as the ratio $\mu^{(\nu)}_{j,j(q_\nu-1)+1}/N^j$ for $N=500$ as a function of ν. It is clearly observed the expected asymptotic limit

$$\lim_{\nu\to\infty} A_{j,q_\nu} = \frac{1}{2^j}. \tag{24}$$

A further generalization for the scaling of the Q-moments must be done. Not shown calculations reveal that relation (23) still holds when $Q=j(q-1)+1$ and q takes on *any* real value, not necessarily the values indicated in Table I (which, in turn, implies that Q may take on any real value). Thus, it can be stated that

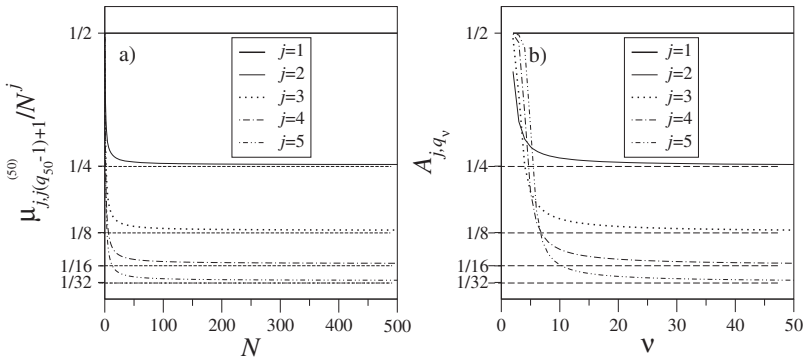

FIG. 5. (a) Ratio $\mu^{(\nu)}_{j,j(q_\nu-1)+1}/N^j$ with $\nu=50$ as a function of N and $j=1$, 2, 3, 4, and 5 from top to bottom. (b) A_{j,q_ν}, calculated as the ratio $\mu^{(\nu)}_{j,j(q_\nu-1)+1}/N^j$ with $N=500$, as a function of ν and the same values of j as in (a).

$$\mu^{(\nu)}_{j,j(q-1)+1} \sim A_{j,q,\nu} N^j, \quad \forall\, q, \quad \forall\, \nu, \tag{25}$$

which means that the whole family of triangles (11) or, equivalently, (15) share the same scaling behavior for their escort moments $\mu^{(\nu)}_{j,Q}$.

V. SCALING OF THE UNNORMALIZED *q*-CUMULANTS

We shall now turn to the study of the unnormalized q-cumulants $\kappa'_{j,q}$ given in Eq. (10). Inasmuch the unnormalized q-cumulant on the order of j, $\kappa'_{j,q}$, is a linear combination of products of unnormalized Q-moments, which is homogeneous on the order j of the q-cumulant [see restriction $\Sigma_{i=1}^{j} i n_i = j$ in the sum defining the unnormalized q-cumulant in Eq. (10)], the scaling of $\kappa'_{j,q}$ will coincide with the scaling of the unnormalized Q-moment $\mu'_{j,Q}$ with $Q=j(q-1)+1$.

Figure 6 shows the asymptotic behavior of the unnormalized escort moments $\mu'^{(\nu)}_{j,j(q-1)+1}$ for the $\nu=3$ ($q_3=1/2$, $\bar{q}_3=9/7$) triangle for $q=3/2$ (left) 2 (center) and 5/2 (right). The following scaling relation is fulfilled:

$$\mu'^{(\nu)}_{j,j(q-1)+1} \sim A'_{j,q,\nu} N^{j(2-q)}. \tag{26}$$

Thus, the independence of the value of q characterizing the scaling exponent of the escort moments [see Eq. (25)] no longer holds for the unnormalized escort moments. Some comments must be made on the validity of scaling law (26). It is valid for any positive q for $\nu \geq 1$ (though it may fail for q close to 0 and values of $j>1$). In the case $\nu<1$, Eq. (26) holds only for $q \leq 1$, while the exponent scaling depends on the value of $\nu<1$ in a complicated fashion for $q>1$. It must be said, however, that relation (26) is only approximated when q approaches 2. For $q=2$, in the transition

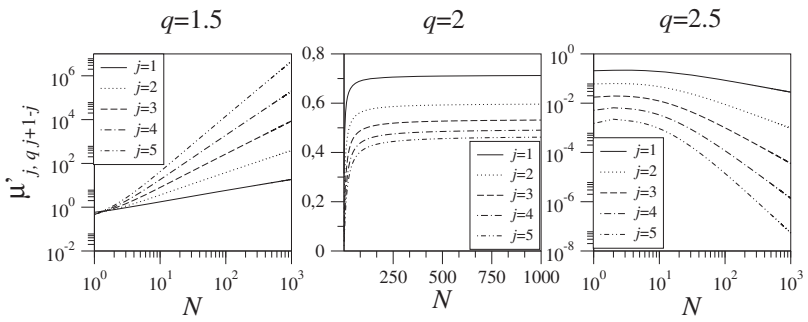

FIG. 6. Scaling of the unnormalized escort moments $\mu'^{(\nu)}_{j,j(q-1)+1}$ for the triangle $\nu=3$ ($q_3=1/2, \bar{q}_3=9/7$) for $q=3/2$ (left) 2 (center) and 5/2 (right) for $j=1-5$. The scaling law (26) is obtained.

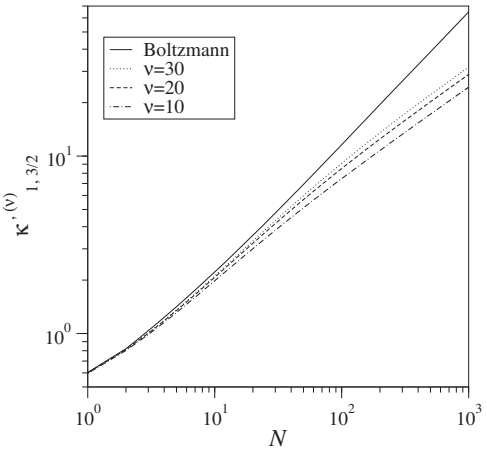

FIG. 7. Scaling of $\kappa'^{(\nu)}_{1,3/2}=\mu'^{(\nu)}_{1,3/2}$ for $\nu=10$, 20, 30, and ∞. The scaling exponent is 1/2 for $\nu<\infty$ and 3/4 for $\nu=\infty$.

from the increasing to the decreasing trend of $\mu'^{(\nu)}_{j,Q}$ with N, a slight increase of the unnormalized Q-moments with N is observed (see Fig. 6) instead of the expected constant value. We will discuss this point in detail later when studying the scaling of the unnormalized q-cumulants. Finally, for $q=1$ the unnormalized moments reduce to the standard ones and relations (25) and (26) coincide.

By comparing Eqs. (16), (25), and (26), the scaling law for the normalizing coefficients of the escort distributions $\gamma^{(\nu)}_{Q}$ is readily deduced as

$$\gamma^{(\nu)}_{j(q-1)+1} \sim \frac{A'_{j,q,\nu}}{A_{j,q,\nu}} N^{j(1-q)} \tag{27}$$

within the same range of parameters as stated above. For the $q=1$ case, Eq. (27) is trivially fulfilled, representing the normalization of the family of probability distributions (13) as $\gamma^{(\nu)}_{1}=\Sigma_{n=0}^{N} p^{(\nu)}_{N,n}=1$.

As already mentioned, an analogous relation to (26) holds for the unnormalized q-cumulants,

$$\kappa'^{(\nu)}_{j,q} \sim C_{j,q,\nu} N^{j(2-q)}, \tag{28}$$

with the same restrictions for the values of q and ν. Special cases are $q=0$, for which *all* the unnormalized 0-cumulants vanish except for the first one [see Eq. (9)], which coincides with the corresponding unnormalized escort moment $\kappa'^{(\nu)}_{1,0}=\mu'^{(\nu)}_{1,0} \propto N^2$, and $q=1$, for which the standard cumulants are recovered as well as an analogous scaling law as that for the Leibnitz case (22) for any value of ν, that is, $\kappa^{(\nu)}_{j,1} \propto N^j$ with vanishing odd cumulants from the third on.

Nevertheless, relations (26)–(28) are no longer valid in the $\nu \to \infty$ limit, that is, for the binomial distribution. It may be shown that in this case the scaling for $N \to \infty$ is

$$\mu'^{(\infty)}_{j,j(q-1)+1} \propto \kappa'^{(\infty)}_{j,q} \propto N^{j(3-q)/2}, \quad \gamma^{(\infty)}_{j(q-1)+1} \propto N^{j(1-q)/2}, \tag{29}$$

which coincide with relations (26)–(28) for $q=1$. The very slow and nonuniform character of the convergence in $\lim_{\nu\to\infty} r^{(\nu)}_{N,n}=1/2^N$ and the numerical difficulties in dealing with coefficients $r^{(\nu)}_{N,n}$ for large ν and N make it subtle to study in more detail the transition of the scaling exponents from the $\nu<\infty$ to the Boltzmann–Gibbs case. Figure 7 shows the scaling of the first unnormalized 3/2-cumulant for triangles with $\nu=10$, 20, and 30, together with the Boltzmann–Gibbs case. The crossover from the scaling exponent 1/2 given by (28) for $\nu<\infty$ and exponent 3/4 indicated in (29) for $\nu=\infty$ is clearly observed. Finally, Eq. (29) is valid for $q \geq 1$ for the unnormalized escort moments $\mu'^{(\infty)}_{j,j(q-1)+1}$ and their corresponding normalizing coefficients $\gamma^{(\infty)}_{j(q-1)+1}$. However, for the unnormalized q-cumulants $\kappa'^{(\infty)}_{j,q}$, its validity restricts to $q>1$. The case $q=1$ must be excluded

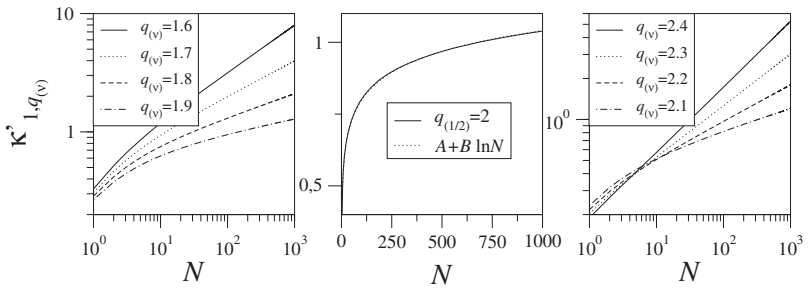

FIG. 8. log-log plots of $\kappa'^{(\nu)}_{1,q_{(\nu)}}=\mu'^{(\nu)}_{1,q_{(\nu)}}$ vs N for $q_{(\nu)}=\bar{q}_\nu=1.6$ to $\bar{q}_\nu=1.9$ (left) and $\bar{q}_\nu=2.1$ to $\bar{q}_\nu=2.4$ (right). The center panel shows the $\bar{q}_{1/2}=2$ case and its fitting to a $A+B\ln N$ curve with $A=0.315$ and $B=0.105$. Both curves virtually overlap.

since the 1-cumulants (normalized or not) coincide with the standard ones, whose scaling $\kappa^{(\infty)}_{j,1}\propto N$ has been already obtained in (20).

Let us connect now the values of ν and q in Eq. (28), i.e., let us study, for the ν triangle (11), the q-cumulants with q being the corresponding limiting value of the q-Gaussian (14). As stated in Table I, there is only one limiting value, $q_{\lim}=\bar{q}_\nu$ for $\nu\in(0,1)$, but two different options, namely, $q_{\lim}=q_\nu$ and $q_{\lim}=\bar{q}_\nu$, for $\nu\geq 1$. We shall denote with $q_{(\nu)}$ any of them and calculate the unnormalized q-cumulant $\kappa'^{(\nu)}_{j,q_{(\nu)}}$.

Following the comments made for the validity of Eqs. (26)–(28), we may substitute q by $q_{(\nu)}$ in Eq. (28) if $\nu\geq 1$ since it is valid for any value of q. The $\nu<1$ case must be studied separately since for this case $q_{(\nu)}=\bar{q}_\nu>1$ and, as already mentioned, Eq. (28) no longer holds. Nevertheless, provided $1/2<\nu<1$, Eq. (28) still holds but now only for the specific value $q=q_{(\nu)}$. On the contrary, for $\nu<1/2$ (hence, $\bar{q}_\nu>2$), the scaling exponent reverts its sign. Summarizing, the following scaling law is obtained

$$\kappa'^{(\nu)}_{j,q_{(\nu)}}\sim C_{j,q_{(\nu)}}N^{j|2-q_{(\nu)}|}. \tag{30}$$

Some comments on the validity of relation (30) must be made. As shown in Fig. 8, for $\bar{q}_{1/2}=2$, a logarithmic increase with N of $\kappa'^{(1/2)}_{1,2}=\mu'^{(1/2)}_{1,2}$ is obtained instead of the expected constant value predicted by (30), which therefore is no longer valid in this case and has to be replaced by a logarithmic law $\kappa'^{(1/2)}_{1,2}\sim A+B\ln N$, whose coefficients are found to be $A=0.315$ and $B=0.105$. This logarithmic correction influences numerically the scaling relation (30) for $q_{(\nu)}$ close to 2, as can be seen in Fig. 8. Finally, for the $q<0$ case, the scaling law still works (30) for $j=1$. Figure 9 shows the actual value of the scaling exponent $\alpha_{1,q_{(\nu)}}=|2-q_{(\nu)}|$ of the first-order unnormalized q-cumulant $\kappa'^{(\nu)}_{1,q_{(\nu)}}=\mu'^{(\nu)}_{1,q_{(\nu)}}\sim C_{1,q_{(\nu)}}N^{\alpha_{1,q_{(\nu)}}}$. A small deviation from the scaling law (30) is observed for $q_{(\nu)}>2$.

Concerning the proportionality coefficient in (30), it must be noticed that

$$\lim_{\nu\to\infty}C_{j,q_{(\nu)}}=\frac{\delta_{1j}}{2}, \tag{31}$$

so the dominant term in the scaling law (30) vanishes for $j>1$. Subdominant terms are also negligible, but in the limit $\nu\to\infty$, a dominant linear term arises so one recovers the extensive behavior $\kappa^{(\infty)}_{j,1}\sim N$ advanced in (20).

The left panel of Fig. 10 shows the order 1 $q_{(\nu)}$-cumulants in the Boltzmann–Gibbs limit $\nu\to\infty$. The predicted trend $\lim_{\nu\to\infty}\kappa'^{(\nu)}_{1,q_{(\nu)}}=\kappa'^{(\infty)}_{1,1}=N/2$ is clearly observed. The right panel of Fig. 10 shows the limit (31) for $j=1$ and 2. (Notice that Figs. 7 and 10 differ in the fact that for Fig. 7 we take the $\nu\to\infty$ limit for a q-cumulant with a fixed $q=3/2$ value, whereas in Fig. 10 we take the same $\nu\to\infty$ limit but for the corresponding $q_{(\nu)}$-cumulant, so $q_{(\nu)}\to 1$).

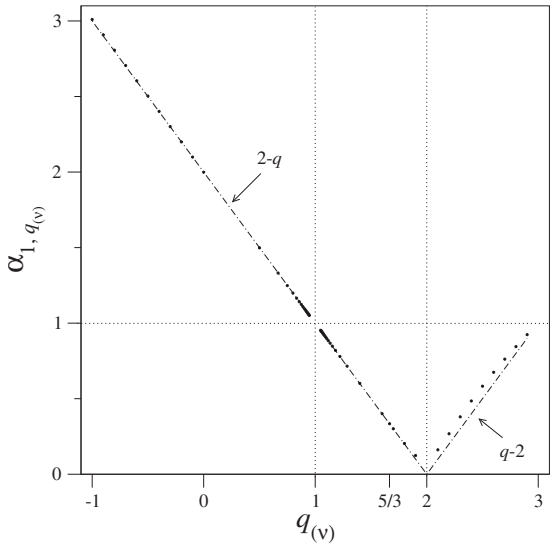

FIG. 9. Scaling exponent $\alpha_{1,q_{(\nu)}}$ of the first-order unnormalized q-cumulant $\kappa'^{(\nu)}_{1,q_{(\nu)}}=\mu'^{(\nu)}_{1,q_{(\nu)}}\sim C_{1,q_{(\nu)}}N^{\alpha_{1,q_{(\nu)}}}$. A small deviation from the scaling law $\alpha_{1,q_{(\nu)}}=|2-q_{(\nu)}|$ given in Eq. (30) is observed for $q_{(\nu)}>2$. Exponents have been obtained via linear regression of the $\ln(\kappa'^{(\nu)}_{1,q_{(\nu)}})$ vs $\ln N$ curves for N from 500 to 1000. The $|q-2|$ curve is also plotted as a guide to the eye.

VI. CONCLUSION

We have studied in detail the scaling with the system size of the Q-moments as well as the q-cumulants—normalized or not—of the family of scale-invariant triangles introduced in Sec. III. We summarize the scaling laws that we have established in Table III. The prefactors have been indicated whenever known. For the range of validity of expressions in the last column of the table, see Sec. V. The scalings with N of the $\nu<\infty$ column reduce to the $\nu\to\infty$ column for $q=1$ *except* for the unnormalized q-cumulants for $j>1$. This unexpected feature (see Fig. 7), i.e., the fact that the $q\to 1$ and $N\to\infty$ limits do not commute, constitutes an interesting result of the present paper.

We believe that the present generalization of the cumulant expansion could be of interest for analytically discussing long-range-interacting systems (e.g., Refs. 28 and 29), for which moments above a given order typically diverge.

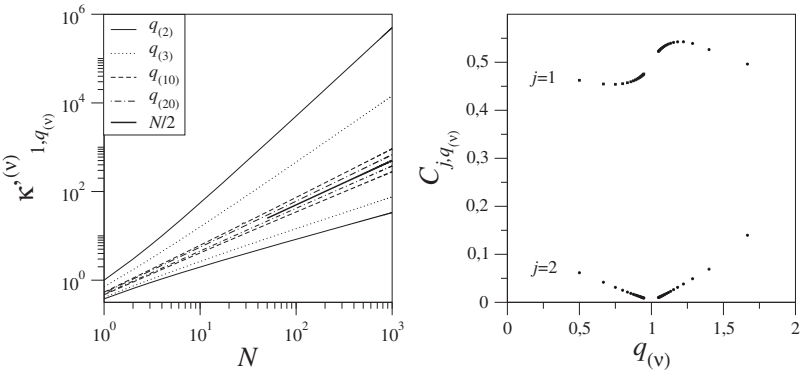

FIG. 10. (Left) log-log plot of $\kappa'^{(\nu)}_{1,q_{(\nu)}}=\mu'^{(\nu)}_{1,q_{(\nu)}}$ vs N for conjugated pairs of triangles $\nu=2$, 3, 10, and 20. The limiting curve $\kappa^{(\infty)}_{1,1}=N/2$ is plotted as a guide to the eye. (Right) Proportionality coefficient $C_{j,q_{(\nu)}}$ in (30) for $j=1$ and 2 as a function of $q_{(\nu)}$. The expected limit (31) is obtained.

TABLE III. Unnormalized Q-moments, normalization factors, ordinary Q-moments and unnormalized q-cumulants for de Boltzmann–Gibbs case (third column), and the family of triangles (fourth column) for $j=1$, 2 and the general case. In all cases, the symbols $\sim$ and $\propto$ are to be understood in the $N\to\infty$ limit.

		$\nu\to\infty$ $(Q=q=1)$	$\nu<\infty$ $(Q=j(q-1)+1)$
$j=1$	$\mu'^{(\nu)}_{1,Q}$	$N/2$	$\propto N^{(2-q)}$
	$\gamma^{(\nu)}_{Q}$	1	$\propto N^{(1-q)}$
	$\mu^{(\nu)}_{1,Q}=\mu'^{(\nu)}_{1,Q}/\gamma^{(\nu)}_{Q}$	$N/2$	$N/2$
	$\kappa'^{(\nu)}_{1,q}$	$N/2$	$\propto N^{(2-q)}$
$j=2$	$\mu'^{(\nu)}_{2,Q}$	$\sim N^2/4$	$\propto N^{2(2-q)}$
	$\gamma^{(\nu)}_{Q}$	1	$\propto N^{2(1-q)}$
	$\mu^{(\nu)}_{2,Q}=\mu'^{(\nu)}_{2,Q}/\gamma^{(\nu)}_{Q}$	$\sim N^2/4$	$\propto N^2$
	$\kappa'^{(\nu)}_{2,q}$	$N/4$	$\propto N^{2(2-q)}$
j	$\mu'^{(\nu)}_{j,Q}$	$\sim N^j/2^j$	$\propto N^{j(2-q)}$
	$\gamma^{(\nu)}_{Q}$	1	$\propto N^{j(1-q)}$
	$\mu^{(\nu)}_{j,Q}=\mu'^{(\nu)}_{j,Q}/\gamma^{(\nu)}_{Q}$	$\sim N^j/2^j$	$\propto N^j$
	$\kappa'^{(\nu)}_{j,q}$	$\sim N$	$\propto N^{j(2-q)}$

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APPENDIX: *q*-EXPONENTIAL AND *q*-LOGARITHM FUNCTIONS

For $q\in\mathbb{R}$, $q\neq 1$, the q-exponential function and its inverse, the q-logarithm function, are defined as

$$e_q^x \equiv [1+(1-q)x]^{1/(1-q)}, \tag{A1}$$

$$\ln_q x \equiv \frac{x^{1-q}-1}{1-q}, \tag{A2}$$

where the standard logarithm and exponential functions are recovered in the limit $q\to 1$. Their respective Taylor expansions are given by (see, for instance, the last of Refs. 4)

$$e_q^x=\sum_{n=0}^{\infty}\frac{e(q,n)}{n!}x^n, \quad e(q,n)=\begin{cases}1, & n=0,1\\ q(2q-1)\cdots((n-1)q-n+2), & n>1,\end{cases}$$

$$\ln_q(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}l(q,n)x^n, \quad l(q,n)=\begin{cases}1, & n=1\\ q(q+1)\cdots(q+n-2), & n>1.\end{cases} \tag{A3}$$

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