Classical field theory for a non-Hermitian Schrödinger equation with position-dependent masses

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A linear one-dimensional Schrödinger equation, defined by means of a non-Hermitian Hamiltonian characterized by position-dependent masses, was proposed lately. Herein we present an exact classical field theory for this equation, showing the need for an extra field $\Phi(x,t)$, in addition to the usual one, $\Psi(x,t)$, similar to what was done recently in the analysis of a class of nonlinear quantum equations. These generalizations of the Schrödinger equation depend on an index q, in such a way that the standard case is recovered in the limit $q \to 1$. Particularly, the field $\Phi(x,t)$ becomes $\Psi^*(x,t)$ only when $q \to 1$ and satisfies a similar Schrödinger equation for the Hermitian conjugate of the Hamiltonian operator. In terms of these two fields one may define a probability density following a standard continuity equation, leading to the preservation of probability in Cartesian space. Simple applications are performed by solving the equations for the two fields.

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I. INTRODUCTION

The constant-coefficient linear differential equations of physics are known nowadays to be applicable mostly to idealized systems, being valid for media characterized by specific conditions, such as homogeneity, isotropy, and translational invariance, with particles interacting through short-range forces and with a dynamical behavior described by short-time memories. However, many real systems—specially the ones within the realm of complex systems—do not fulfill these requirements, e.g., one may have a breakdown of translation, and/or rotational invariance, with long-range interactions among particles, which may be also immersed in some type of nonlinear media, and so on. In these cases one needs to introduce modifications in such equations, by considering, e.g., position-dependent coefficients, or nonlinear terms, which may lead among many aspects, to non-Hermitian Hamiltonians [1–5]. As a consequence, finding analytical solutions may become a hard task, and very frequently one has to make use of numerical procedures. Due to the latest advances in computer technology, the study of such generalized equations has gained a lot of interest and a considerable advancement has been attained, leading to progress in many areas of physics that use these types of equations, such as nonlinear optics, superconductivity, plasma physics, and nonequilibrium statistical mechanics.

Certainly, the Schrödinger equation (SE) represents one of the most important equations of physics and it has been modified in recent years in order to take into account such effects (see, e.g., Refs. [4,6–16]). Most of these modifications appear as generalizations of the standard linear equation, so that this equation may be recovered as particular cases. Among many possible ways, the following procedures have been performed in the literature lately: (i) addition of new nonlinear terms to the linear equation; (ii) modification of exponents of existing linear terms; (iii) introduction of position-dependent coefficients. One should mention that the first two cases lead to different types of nonlinear equations, whereas a linear equation follows in the third case. A well-known type of nonlinear SE was proposed a few decades ago, according to the first procedure [4]: the introduction of an extra cubic term in the wave function is responsible for the modulation of the wave function for some particular type of solution. The second scheme was applied in Refs. [7–9] and it has been much used within nonextensive statistical mechanics [17]; this theory emerged from the generalization of the Boltzmann-Gibbs entropy, by introducing a real index q, such as to recover the former in the limit $q \to 1$ [18]. In case (iii) one should mention the proposals of Refs. [6,10–16], which considered a linear SE with position-dependent masses, motivated by an appropriate description of semiconductor heterostructures [20]. Herein, we will be mostly interested in the recent work of Ref. [14] and its extensions [15,16], where a displacement operator was defined in terms of a deformed derivative associated with nonextensive statistical mechanics, also leading to a linear SE with position-dependent masses.

In a recent work [8], a classical field theory for the nonlinear SE introduced in Ref. [7] was formulated. It was shown that besides the usual $\Psi(\vec{x},t)$, a new field $\Phi(\vec{x},t)$ has to be considered in order to fulfill the equations of motion; this new field, which satisfies an additional equation, becomes $\Psi^*(\vec{x},t)$ only when $q \to 1$. Herein we will present a similar approach for the one-dimensional SE introduced in Ref. [14], namely,

$$\begin{aligned} i\hbar \frac{\partial \Psi(x,t)}{\partial t} &= -\frac{\hbar^2}{2m} \hat{D}_\gamma^2 \Psi(x,t) + V(x)\Psi(x,t), \\ \hat{D}_\gamma &= (1+\gamma x)\frac{d}{dx}, \end{aligned} \tag{1}$$

where $V(x)$ denotes a potential and $\hat{D}_\gamma$ is a deformed derivative in space [19], associated with the deformed momentum operator $\hat{p}_\gamma = -i\hbar \hat{D}_\gamma$. This equation may be written also in terms of a Hamiltonian operator $\hat{H}$,

$$\begin{aligned} i\hbar \frac{\partial \Psi(x,t)}{\partial t} &= \hat{H}\Psi(x,t), \\ \hat{H} &= \frac{1}{2m}(1+\gamma x)\hat{p}(1+\gamma x)\hat{p} + V(x), \end{aligned} \tag{2}$$

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where $\hat{p} = -i\hbar(d/dx)$ stands for the usual momentum operator of quantum mechanics. One readily sees that the deformed momentum operator is non-Hermitian, i.e., $\hat{p}_\gamma^\dagger \neq \hat{p}_\gamma$ and consequently, $\hat{H}^\dagger \neq \hat{H}$ [15,16].

The parameter γ has been related to a deformation in space and can be identified as $\gamma = (1-q)/L_0$, where L_0 represents a characteristic length of a given physical system, and q is the entropic index of nonextensive statistical mechanics [14]; by means of this identification, the resulting deformed-derivative operator $\hat{D}_q$ was shown to be very useful for performing calculations in this theory [19]. Alternatively, γ may also turn up in the definition $m_e = m/(1+\gamma x)^2$, interpreted as the particle's position-dependent effective mass, found in real systems, as in semiconductors of nonuniform chemical composition [6] and semiconductor heterostructures [20]. In terms of this effective mass, Eq. (1) [or equivalently, Eq. (2)] becomes

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m_e}\frac{\partial^2\Psi(x,t)}{\partial x^2} - \frac{\hbar^2}{2}\left[\frac{d}{dx}\left(\frac{1}{2m_e}\right)\right]\frac{\partial\Psi(x,t)}{\partial x} + V(x)\Psi(x,t). \tag{3}$$

The difficulty of the non-Hermitian Hamiltonian was circumvented in Ref. [15] by redefining the deformed momentum operator, leading to the Hermitian Hamiltonian

$$\hat{H}' = \frac{1}{2m}(1+\gamma x)\hat{p}(1+\gamma x)\hat{p} + \frac{i\hbar\gamma}{2m}(1+\gamma x)\hat{p} - \frac{\hbar^2\gamma^2}{8m} + V(x). \tag{4}$$

Clearly, the Hamiltonian $\hat{H}'$ is associated with a different physical system, as one sees by comparing Eqs. (2) and (4), which should lead, respectively, to distinct energy spectra and expectation values. However, nowadays it is known that Hermiticity is only sufficient, but not a necessary condition for a consistent quantum theory, since it has been demonstrated in the literature that non-Hermitian Hamiltonians may also present real energy eigenvalues, leading to a well-defined quantum theory [5,21–26]. In the present work we reinforce this assertion by showing that the problem described by the SE of Eq. (2) is also well defined, since it follows a standard continuity equation for a general potential $V(x)$, and that the energy eigenvalues may be positive definite in the stationary state. Hence, herein we will continue the analysis of Ref. [14], by discussing further the non-Hermitian Hamiltonian of Eq. (2). To present a consistent quantum mechanics from this Hamiltonian and to find a recipe of how to compute expectation values of physical quantities, we construct a classical field theory for this SE in the next section. For this purpose, we must introduce an extra field $\Phi(\vec{x},t)$, similar to what was done in Ref. [8] for the nonlinear SE proposed in Ref. [7]. Two simple standard examples are considered, namely, a particle in a constant potential and the one of an infinite square well; in these examples we solve the equations for both fields, showing that the energy eigenvalues are real. In Sec. III we derive the classical Hamiltonian density and the standard continuity equation. Finally, in Sec. IV we present our main conclusions.

II. CLASSICAL FIELD THEORY

We will now develop an exact classical field theory associated with Eq. (2) and its complex conjugate. For reasons that will become clear later, we introduce also an additional dimensionless field, $\Phi(\vec{x},t)$. The equations of motion for classical fields can be derived by means of the principle of stationary action, through the definition of a Lagrangian density $\mathcal{L}$, which will depend on the dimensionless fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$, on their time derivatives, as well as spatial derivatives,

$$\mathcal{L} \equiv \mathcal{L}\big(\Psi,\partial_x\Psi,\partial_x^2\Psi,\partial_t\Psi,\Phi,\partial_x\Phi,\partial_t\Phi, \Psi^*,\partial_x\Psi^*,\partial_x^2\Psi^*,\partial_t\Psi^*,\Phi^*,\partial_x\Phi^*,\partial_t\Phi^*\big), \tag{5}$$

where $\partial_x \equiv \partial/\partial x$, $\partial_x^2 \equiv \partial^2/\partial x^2$, and $\partial_t \equiv \partial/\partial t$. Now we consider heuristically the Lagrangian density,

$$\begin{aligned}\mathcal{L} = {} & \frac{i\hbar}{2}\Phi(x,t)\partial_t\Psi(x,t) + \frac{\hbar^2}{8}\left[\frac{d}{dx}\left(\frac{1}{m_e}\right)\right]\Phi(x,t)\partial_x\Psi(x,t) \\ & + \frac{\hbar^2}{4m_e}\Phi(x,t)\,\partial_x^2\Psi(x,t) - \frac{1}{2}V(x)\Psi(x,t)\Phi(x,t) \\ & - \frac{i\hbar}{2}\Phi^*(x,t)\partial_t\Psi^*(x,t) \\ & + \frac{\hbar^2}{8}\left[\frac{d}{dx}\left(\frac{1}{m_e}\right)\right]\Phi^*(x,t)\partial_x\Psi^*(x,t) \\ & + \frac{\hbar^2}{4m_e}\Phi^*(x,t)\,\partial_x^2\Psi^*(x,t) - \frac{1}{2}V(x)\Psi^*(x,t)\Phi^*(x,t).\end{aligned} \tag{6}$$

One should notice that the Lagrangian density defined above presents a dependence up to the first spatial derivative in the field $\Phi(x,t)$ and up to the second one in the field $\Psi(x,t)$; this represents an important requirement for obtaining the correct Euler-Lagrange equations for these two fields. Therefore, in the first case one has a standard Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_x\left[\frac{\partial\mathcal{L}}{\partial(\partial_x\Phi)}\right] - \partial_t\left[\frac{\partial\mathcal{L}}{\partial(\partial_t\Phi)}\right] = 0, \tag{7}$$

whereas in the second case, one should take into account the contribution from its second-derivative term [27,28],

$$\frac{\partial\mathcal{L}}{\partial\Psi} - \partial_x\left[\frac{\partial\mathcal{L}}{\partial(\partial_x\Psi)}\right] - \partial_t\left[\frac{\partial\mathcal{L}}{\partial(\partial_t\Psi)}\right] + \partial_x^2\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_x^2\Psi\right)}\right] = 0. \tag{8}$$

Substituting the Lagrangian density of Eq. (6) in the Euler-Lagrange of Eq. (7), one recovers the SE in Eq. (3). Carrying the same procedure in the Euler-Lagrange in Eq. (8), one obtains the equation for the field Φ,

$$-i\hbar\frac{\partial\Phi(x,t)}{\partial t} = -\frac{\hbar^2}{2m_e}\frac{\partial^2\Phi(x,t)}{\partial x^2} - \frac{3\hbar^2\gamma(1+\gamma x)}{2m}\frac{\partial\Phi(x,t)}{\partial x} - \frac{\hbar^2\gamma^2}{2m}\Phi(x,t) + V(x)\Phi(x,t), \tag{9}$$

which may be written in the form of Eq. (2), namely,

$$-i\hbar\frac{\partial\Phi(x,t)}{\partial t} = \hat{H}^\dagger\Phi(x,t), \tag{10}$$

where $\hat{H}^\dagger$ is the Hermitian conjugate of the Hamiltonian operator defined in Eq. (2). Therefore, due to the non-Hermiticity

of the Hamiltonian, Eqs. (2) and (10) [or equivalently, Eqs. (3) and (9)] are distinct and not related through a complex-conjugate operation, leading to a field $\Phi(x,t)$ different from $\Psi^*(x,t)$. In principle, the simplest case with the field $\Phi(x,t)$ connected to $\Psi^*(x,t)$ occurs only for $\gamma \to 0$, i.e., $q \to 1$, yielding $\hat{H} = \hat{H}^\dagger$, together with the identification $\Phi(x,t) = \Psi^*(x,t)$. However, as it will be seen next, there are specific situations where $\gamma \neq 0$, but $\Phi(x,t)$ and $\Psi^*(x,t)$ may still follow a simple relation, leading to a type of solution with many potential applications, e.g., in the description of semiconductor heterostructures [14], or a quantum Morse oscillator relevant for diatomic molecules [16]. The case where these two fields are not related through a complex-conjugate operation may still correspond to the physically interesting situation where they can be associated with particles that do not interact with light [8].

Moreover, as it will be shown in the next section, the present two-field formulation yields a standard continuity equation, valid in Cartesian space for all γ, in contrast to the single-field proposal of Refs. [14,16], which applies to a deformed, i.e., dilated ($\gamma > 0$) or contracted ($\gamma < 0$), space. Therefore, within the present framework, the probability is conserved in Cartesian space for a probability density defined as

$$\rho(x,t) = \tfrac{1}{2}[\Psi(x,t)\Phi(x,t) + \Psi^*(x,t)\Phi^*(x,t)]. \tag{11}$$

For a general potential $V(x)$, Eqs. (3) and (9) admit solutions of the type

$$\Psi(x,t) = \exp\left(-\frac{iE_\psi t}{\hbar}\right)\psi(x), \quad \Phi(x,t) = \exp\left(\frac{iE_\phi t}{\hbar}\right)\phi(x), \tag{12}$$

leading to the set of time-independent equations

$$E_\psi \psi = -\frac{\hbar^2}{2m_e}\frac{d^2\psi}{dx^2} - \frac{\hbar^2\gamma(1+\gamma x)}{2m}\frac{d\psi}{dx} + V(x)\psi, \tag{13}$$

$$E_\phi \phi = -\frac{\hbar^2}{2m_e}\frac{d^2\phi}{dx^2} - \frac{3\hbar^2\gamma(1+\gamma x)}{2m}\frac{d\phi}{dx} - \frac{\hbar^2\gamma^2}{2m}\phi + V(x)\phi. \tag{14}$$

Notice that we have used in Eq. (12) a very general type of solution, where each field could, in principle, be associated with a different particle and energy spectrum; the situation $E_\psi = E_\phi = E$ appears as a particular case. Next we will study Eqs. (13) and (14) by considering two simple choices for the potential $V(x)$.

A. Application 1: Constant potential

Let us consider first the simplest case, $V(x) = V_0$ (V_0 positive constant). In this case, the solution of Eq. (13) is given by [14]

$$\psi(x) = C_1 \cos\left[\frac{k_\psi}{\gamma}\ln(1+\gamma x)\right] + C_2 \sin\left[\frac{k_\psi}{\gamma}\ln(1+\gamma x)\right], \tag{15}$$

where $k_\psi = [2m(E_\psi - V_0)]^{1/2}/\hbar$. Furthermore, the solution of Eq. (14) may also be found,

$$\phi(x) = \frac{1}{1+\gamma x}\left\{D_1 \cos\left[\frac{k_\phi}{\gamma}\ln(1+\gamma x)\right] + D_2 \sin\left[\frac{k_\phi}{\gamma}\ln(1+\gamma x)\right]\right\}, \tag{16}$$

with $k_\phi = [2m(E_\phi - V_0)]^{1/2}/\hbar$. If $E_\psi = E_\phi = E$, $D_1 = C_1^*$, and $D_2 = C_2^*$, these two fields are related,

$$\phi(x) = \frac{\psi^*(x)}{1+\gamma x}, \tag{17}$$

such that for $\gamma = 0$, one has $\phi(x) = \psi^*(x)$. However, in the most general case, since we have two distinct fields obeying different equations, their corresponding solutions may not follow a simple relation, like the one above.

B. Application 2: Infinite square well

Let us consider now a particle described by Eqs. (13) and (14) under the potential of an infinite square well, i.e., infinite for $x < 0$ and $x > L$, and zero in the interval $0 < x < L$. Like the standard SE [29], we impose the wave functions $\psi(x)$ and $\phi(x)$ to be zero when the potential is infinite, $\psi(0) = \psi(L) = \phi(0) = \phi(L) = 0$. Considering the simplest case where $E_\psi = E_\phi = E$, one obtains the eigenfunctions,

$$\psi_n(x) = \begin{cases} A_n \sin\left[\frac{k_n}{\gamma}\ln(1+\gamma x)\right], & \text{if} \quad 0 < x < L\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{18}$$

$$\phi_n(x) = \begin{cases} A_n(1+\gamma x)^{-1}\sin\left[\frac{k_n}{\gamma}\ln(1+\gamma x)\right], & \text{if} \quad 0 < x < L, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

To satisfy the boundary conditions, the wave vectors are quantized,

$$k_n = \frac{n\pi\gamma}{\ln(1+\gamma L)}, \quad n = 1,2,\ldots, \tag{20}$$

leading to the eigenvalues

$$E_n = \frac{\hbar^2 n^2 \pi^2 \gamma^2}{2m\ln^2(1+\gamma L)}, \quad n = 1,2,\ldots. \tag{21}$$

One should call attention to the fact that the wave vectors and energy eigenvalues above coincide with those obtained in Ref. [14]. However, as will be seen in the next section, for the conservation of probability in Cartesian space, one needs the additional wave function $\phi_n(x)$ of Eq. (19). In this case, one uses the probability density of Eq. (11) to compute the coefficient A_n that appears in Eqs. (18) and (19), by imposing as usual, $\int_0^L dx\rho(x) = 1$, leading to

$$A_n^2 = \frac{2\gamma}{\ln(1+\gamma L)}, \tag{22}$$

which, in the limit $\gamma \to 0$, leads to $\lim_{\gamma\to 0} A_n^2 = 2/L$.

Since the parameter γ is associated with a deformation in space, leading to non-Hermiticity of the Hamiltonian, it is

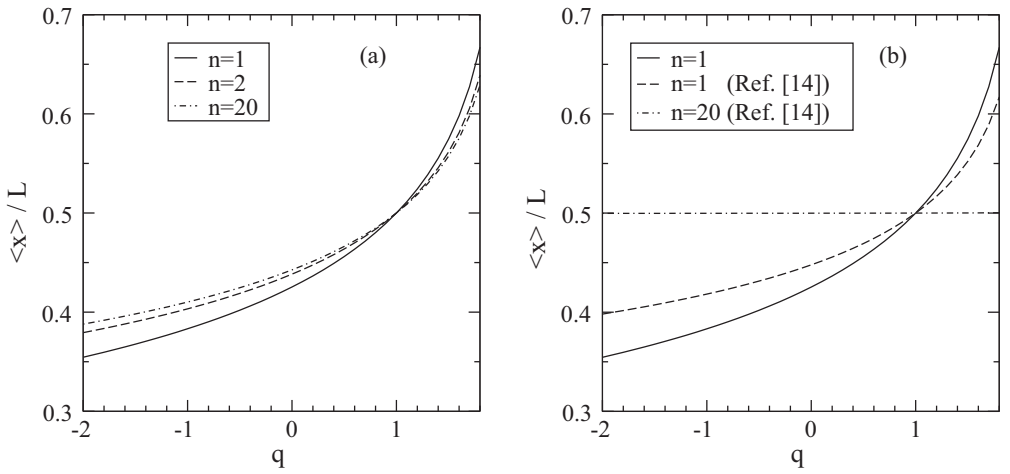

FIG. 1. The particle average position, in units of the size of the square well L, as a function of q. (a) Results from Eq. (25) for typical values of the quantum number n. (b) The result $n = 1$ from Eq. (25) (full line) is compared to those from Ref. [14] for $n = 1$ (dashed line) and $n = 20$ (dotted-dashed line).

important to know how it influences the mean value $\langle x\rangle$:

$$\begin{aligned}\langle x\rangle &= \int_0^L dx\, x\rho(x)\\ &= \frac{1}{2}\int_0^L dx\, x[\psi_n(x)\phi_n(x) + \psi_n^*(x)\phi_n^*(x)]\\ &= \int_0^L dx\, \frac{x}{1+\gamma x}\,\psi_n^2(x). \end{aligned} \tag{23}$$

Using the eigenfunctions in Eqs. (18) and (19),

$$\langle x\rangle = A_n^2\left\{\frac{2Ln^2\pi^2}{\gamma[4n^2\pi^2 + \ln^2(1+\gamma L)]} - \frac{\ln(1+\gamma L)}{2\gamma^2}\right\}, \tag{24}$$

and if one identifies the deformation parameter γ with the entropic index of nonextensive statistical mechanics, $\gamma = (1-q)/L$ [14], the equation above becomes

$$\langle x\rangle = \frac{L}{2}\left\{\frac{8n^2\pi^2}{\ln(2-q)[4n^2\pi^2 + \ln^2(2-q)]} - \frac{2}{1-q}\right\}. \tag{25}$$

The asymmetry in the average position emerges as a consequence of the parameter γ and it disappears, as one can see from Eq. (24) by considering the limit $\gamma \to 0$ [or equivalently, from Eq. (25) by taking the limit $q \to 1$], so that one gets in such limit, $\langle x\rangle \to L/2$.

The mean value of Eq. (25) is exhibited in Fig. 1(a) as a function of q, for typical values of the quantum number n. Two important points may be noticed in Fig. 1(a): (i) In the contracted (dilated) space, which corresponds to $q > 1$ ($q < 1$), the average position of the particle is shifted to values $\langle x\rangle < L/2$ ($\langle x\rangle > L/2$). (ii) The qualitative behavior of $\langle x\rangle$ does not change considerably for increasing values of the quantum number n. In Fig. 1(b) we present the same quantity for the quantum number $n = 1$, comparing it with the calculation carried out in Ref. [14] [cf. Eq. (24) of Ref. [14]], in which a different probability density was considered. Essentially, one notices a significant qualitative discrepancy of the two results for large values of n.

It is important to notice that since A_n is real, Eqs. (18) and (19) yield $\phi_n(x) = \psi_n(x)/(1+\gamma x)(0 < x < L)$, in agreement with Eq. (17) of the previous illustration. The normalization condition becomes, in the present example,

$$\begin{aligned}\int_0^L dx\,\rho(x) &= \int_0^L dx\,\psi_n(x)\phi_n(x)\\ &= \int_0^L dx\,\frac{\psi_n^2(x)}{1+\gamma x} = 1, \end{aligned} \tag{26}$$

which reproduces Eq. (8) of Ref. [16], obtained through a modification of the momentum operator introduced previously in Ref. [14]. This result, derived independently by two different approaches, suggests that the relation of Eq. (17), obtained by the present two-field approach, may be valid in more general situations than those analyzed herein.

In the next section we derive the Hamiltonian density associated with the Lagrangian density of Eq. (6); in addition to this, we show that the probability density of Eq. (11) ensures conservation of probability, by following a continuity equation in Cartesian space.

III. HAMILTONIAN DENSITY AND CONTINUITY EQUATION

Following Ref. [28], let us now define the Hamiltonian density by

$$\begin{aligned}\mathcal{H} = {}& \Pi_\Psi\dot\Psi + \Pi_{\Psi^*}\dot\Psi^* + \Pi_\Phi\dot\Phi + \Pi_{\Phi^*}\dot\Phi^* + \Pi_{\dot\Psi}\ddot\Psi + \Pi_{\dot\Phi}\ddot\Phi\\ &+ \Pi_{\dot\Psi^*}\ddot\Psi^* + \Pi_{\dot\Phi^*}\ddot\Phi^* - \mathcal{L}, \end{aligned} \tag{27}$$

where the canonical momenta conjugated to the fields Ψ, Φ, $\dot\Psi$, and $\dot\Phi$ are

$$\Pi_\Psi = \frac{\partial\mathcal{L}}{\partial\dot\Psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\ddot\Psi} - 2\partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\dot\Psi)}, \tag{28}$$

$$\Pi_\Phi = \frac{\partial\mathcal{L}}{\partial\dot\Phi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\ddot\Phi} - 2\partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\dot\Phi)}, \tag{29}$$

$$\Pi_{\dot\Psi} = \frac{\partial\mathcal{L}}{\partial\ddot\Psi}, \tag{30}$$

$$\Pi_{\dot\Phi} = \frac{\partial\mathcal{L}}{\partial\ddot\Phi}, \tag{31}$$

with similar expressions holding for the corresponding complex conjugates. In the present case, $\Pi_\Psi = i\hbar\Phi/2$, $\Pi_{\Psi^*} = -i\hbar\Phi^*/2$, whereas $\Pi_\Phi = \Pi_{\Phi^*} = \Pi_{\dot\Psi} = \Pi_{\dot\Phi} = \Pi_{\dot\Psi^*} = \Pi_{\dot\Phi^*} = 0$, so that the Hamiltonian density becomes

$$\begin{aligned}\mathcal{H} = &-\frac{\hbar^2}{8}\left[\frac{d}{dx}\left(\frac{1}{m_e}\right)\right]\Phi(x,t)\partial_x\Psi(x,t)\\ &-\frac{\hbar^2}{4m_e}\,\Phi(x,t)\,\partial_x^2\Psi(x,t)+\frac{1}{2}V(x)\Psi(x,t)\Phi(x,t)\\ &-\frac{\hbar^2}{8}\left[\frac{d}{dx}\left(\frac{1}{m_e}\right)\right]\Phi^*(x,t)\partial_x\Psi^*(x,t)\\ &-\frac{\hbar^2}{4m_e}\,\Phi^*(x,t)\,\partial_x^2\Psi^*(x,t)+\frac{1}{2}V(x)\Psi^*(x,t)\Phi^*(x,t).\end{aligned} \tag{32}$$

As a simple example, if one considers solutions like in Eq. (12) in the first application of the previous section, i.e., a particle in a constant potential $V(x) = V_0$ ($V_0 > 0$), the solutions with $E_\psi = E_\phi = E$ yield

$$\mathcal{H} = \frac{\hbar^2k^2}{4m}[\psi(x)\phi(x)+\psi^*(x)\phi^*(x)] = \frac{\hbar^2k^2}{2m}\rho(x), \tag{33}$$

where $k = [2m(E-V_0)]^{1/2}/\hbar$, the functions $\psi(x)$ and $\phi(x)$ are given by Eqs. (15) and (16), and we have used Eq. (11) in the last equality. Thus, for a particle in a box of size L, choosing appropriately the coefficients in Eqs. (15) and (16) so that $\int_0^L dx\rho(x) = 1$, one obtains the energy of the particle under a constant potential:

$$\varepsilon = \int_0^L dx\,\mathcal{H} = \frac{\hbar^2k^2}{2m}. \tag{34}$$

Now we address the important issue of probability conservation, which comes as a consequence of a continuity equation. If one considers the probability density of Eq. (11), together with the equations for the fields [Eqs. (3) and (9)], one verifies a continuity equation for a general potential $V(x)$,

$$\frac{\partial\rho(x,t)}{\partial t}+\frac{\partial j(x,t)}{\partial x} = 0, \tag{35}$$

where

$$\begin{aligned}j(x,t) = \frac{\hbar}{8i}\Bigg\{&\frac{2}{m_e}\,\Phi(x,t)\frac{\partial\Psi(x,t)}{\partial x}-\frac{2}{m_e}\,\Psi(x,t)\frac{\partial\Phi(x,t)}{\partial x}\\ &-\frac{2}{m_e}\,\Phi^*(x,t)\frac{\partial\Psi^*(x,t)}{\partial x}+\frac{2}{m_e}\,\Psi^*(x,t)\frac{\partial\Phi^*(x,t)}{\partial x}\\ &-\left[\frac{d}{dx}\left(\frac{1}{m_e}\right)\right]\Psi(x,t)\Phi(x,t)\\ &+\left[\frac{d}{dx}\left(\frac{1}{m_e}\right)\right]\Psi^*(x,t)\Phi^*(x,t)\Bigg\}.\end{aligned} \tag{36}$$

It is important to recall that in the limit $\gamma = 0$, one has $m_e = m$, $\Phi(x,t) = \Psi^*(x,t)$, and $\Phi^*(x,t) = \Psi(x,t)$, so that the probability density of Eq. (11) and the current density above recover the standard ones [29].

IV. CONCLUSIONS

We have presented an exact classical field theory for a linear one-dimensional Schrödinger equation introduced recently [14], defined in terms of a non-Hermitian Hamiltonian ($\hat H^\dagger \neq \hat H$), associated with position-dependent masses. We have shown the need for an extra field $\Phi(x,t)$ in addition to the usual one, $\Psi(x,t)$, similar to the procedure established in the analysis of a nonlinear Schrödinger equation [8]. We have verified also that in the present case, the field $\Phi(x,t)$ obeys a similar Schrödinger equation, defined for the Hermitian conjugate operator $\hat H^\dagger$. These generalizations of the Schrödinger equation have an aspect in common, namely, that they depend on a real index q in such a way that the standard case is recovered in the limit $q \to 1$. Particularly, the field $\Phi(x,t)$, which satisfies an additional equation, becomes $\Psi^*(x,t)$ only when $q \to 1$. In terms of these two fields, we have defined a probability density that follows a standard continuity equation, leading to the preservation of probability in Cartesian space, and from which one may compute average values. Simple applications of this non-Hermitian Schrödinger equation considering the constant, as well as the infinite square-well potentials, were solved exactly. The present procedure, based on two distinct fields, seems to be quite general and appropriate for a wide class of position-dependent-coefficient, and/or nonlinear quantum equations.

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