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Interplay between polarisation and plurality in a decision-making process with continuous opinions

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Abstract. By considering that the stance held by an individual in a collective decision-making process is intimately related to a quantifiable trait of personality, we study the impact of different sorts of inhomogeneity—namely, polarisation (partisanship) and plurality—of that trait among the population on the final collective stance.

It is shown that, although both situations represent forms of diversity in the system, each one is associated with a different type of emergence of consensus that herein we define as a nontrivial collective stance. Specifically, we verify that plurality only relates to a continuous emergence of nontrivial stances, whereas for given degrees of partisanship a discontinuous transition can be observed. That discontinuity implies the existence of a latent heterogeneity in the system; latent in the sense that it is unable to enhance the decrease in the cost of assuming the trivial collective stance over a nontrivial stance.

Moreover, in bringing forth the existence of a discontinuous transition—and hence of metastable states of consensus—it is possible to adjust partisanship and heterogeneity in the group so that, by increasing the diversity of the relevant trait to the problem, the system can move from trivial to nontrivial stances. Such effect ultimately assigns to diversity, namely polarisation, a counter-intuitive role of consensus provider.

Keywords: critical phenomena of socio-economic systems, interacting agent models, stochastic processes

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Contents

1. Introduction

Whatever its ilk, communities constantly face decision-making processes where a consensus regarding the possibilities that have been tabled is wished for. If for a run-off a two state model is perfectly adequate [1, 2][1], there is a wide variety of cases that do not fit to the existence of only two options; for instance, the decision of economical agents concerning the placement of their orders in an auction; the definition of penalties or duties & rights in an Act or Charter. These problems traditionally give rise to a gamut of possibilities which permit to an element of the community involved in the decision-making process to express her opinion taking into best account her traits—grade of bullishness/bearishness, degree of alignment with right/left-wing policies or whatever is relevant to the case—and that clearly affects the dynamics towards the final decision reached by the group [4–6].

Aiming at coping with such type of problems, i.e. the dynamics of opinions and the achieving (or not) of a consensus in collective social or economical/financial decision-making processes, physicists as well as other researchers have been using tools and techniques originally introduced within a Statistical Physics context [1, 7, 10–12] and which at the end of the day led to new research fields often dubbed sociophysics [1, 7] and econophysics [8, 9].

In opinion models, the qualitative traits of the individuals are quantitatively represented by some parameter (or a set thereof), α, and its diversity is frequenly characterised by assigning the Gaussian or the Uniform distribution to that quantity [13]. Nonetheless, lifelike heterogeneity is certainly better represented by a mixing between polarisation [14]—that we can understand as a proxy for partisanship—and a given

[1] In that case, we could also think of a three state model with the third state representing abstention. Yet, although abstentionists are the main target in political marketing they are scrapped from the electoral results in almost every country and distributed according to their proneness in the polls [3].

level of inhomogeneity around polarised stances that can be linked to plurality [15]. As an example, a politicised society tends to roughly split between left-wing and right-wing individuals. Yet, when we look for instance at the latter, we learn that it actually ranges from ultra-capitalism to social-liberalism [16]. Therefore, it is plausible to try to assess what can be the role of each kind of diversity in the process of collective decision-making and whether they can influence the achievement of consensus and the path towards to that state as well.

That is the goal of the present manuscript. Technically, we do so by employing a self-consistent treatment of the model because (i) it allows analytical—hence more controlled—assertions and (ii) in many situations related to those previously mentioned herein, the agents in the system interact in a global way naturally accommodating a self-consistent approach. It should be noticed that, on the one hand, when we look at global interaction as a mean-field approach, it is possible to find excellent results in the literature regarding theoretical opinion models [17], especially concerning the determination/estimation of their critical point(s); on the other hand, the effective behaviour of a system might be different from the mean-field picture, particularly when the models are defined on networks with complex topology or in field studies [18]. Still, that sort of approach tends to yield relevant ballpark figures for the critical parameters and also indications to further improvements in modeling that are usually just solved by numerical simulation.

The remaining of this manuscript is organised as follows: in section 2 we introduce our model providing its generic analytical treatment; in section 3 we present and analyse the main core of our results, particularly the phase diagrams in the self-consistent approach; finally we convey our final remarks and perspectives of future work in section 4. Last, it is worth referring that in appendix are set forth some scenarii to appraise the validity of the model, including the determination of parameter α in social systems.

2. The model

Let us consider a system composed of N individuals, $\{i\}$, each one expressing her *stance* (or opinion), s_i, that is represented by a real number. The overall state of the community, $\mathbf{s} \equiv \{s_i\}$, is governed by the funcion[2],

$$U_T(\mathbf{s}) \equiv \sum_i [\, U_i^{\text{end}}(\mathbf{s}) + U_i^{\text{int}}(\mathbf{s})], \tag{1}$$

that we understand as a cost function. The variable s is assumed as generic and acquires a specific meaning depending on the problem under debate. In other words, it can be a political, economical or financial position which is manifestly ruled by some angle of personality of the agent, e.g. the degree of risk aversion, orientation regarding political views, beliefs among others.

The term U_i^{end} is defined as endogenous and goes as follows: first, we have a contribution that intends to represent the average trait of the individuals—e.g. on average

[2] Often the value of the opinion is bounded, but in this case we let it assume any real value aiming to enclose the representation of extremist opinions as well.

humans are prone to risk aversion—and that paves the way to the emergence of a final position related to a global state that most conveniently and generically attends to whole group, i.e. the *optimal collective stance*. Concomitantly, it must be taken into consideration that the diversity in the trait relevant to the decision-making process changes what each individual considers the optimal stance. For instance, a greedy investor will tend to claim that more than the widely mentioned '10% of the savings' rule-of-thumb [19] should be invested in the stock market. Bringing these points together we have

$$U_i^{\mathrm{end}}(s) \equiv \frac{a}{2} s_i^2 + \frac{b}{4} s_i^4 - \alpha_i\, s_i, \tag{2}$$

where $b > 0$, a is a real value. The trait of each individual is reflected in the value of the parameter α_i and the values of a and b represent common (average) features within the community. In real situations this could be assigned to (average) age, (majority) gender, education or even geographical origin.

As aforementioned, opinion models traditionally centre attention in the plurality and assume Gaussian/Uniform distributions to represent it [13]. In our case, we impose α quenched[3] and obeying a double Gaussian,

$$p(\alpha) = \frac{1}{\sqrt{8\pi}\,\sigma} \sum\nolimits_{\pm} \left\{ \exp\left[-\frac{(\alpha \pm \mu)^2}{2\,\sigma^2} \right] \right\}, \tag{3}$$

that has a very flexible structure. Specifically, probability density function (3) converts into a single peaked distribution when $\mu < \sigma$ and to the bimodal distribution when $\sigma \to 0$, $p(\alpha) = \frac{1}{2}\delta(\alpha - \mu) + \frac{1}{2}\delta(\alpha + \mu)$. Regarding its moments, the distribution has average, $\langle \alpha \rangle = 0$—as well as all its odd moments—and variance $\langle \alpha^2 \rangle = \sigma_\alpha^2 = \mu^2 + \sigma^2$. Distribution (3) is platykurtic with the kurtosis, $\kappa \equiv 3 - \frac{2\mu^4}{(\sigma_\alpha^2)^2}$, less than 3 for all $\mu \neq 0$. Figure 1 shows representative cases as well as specific distributions that are relevant to our results.

Let us look at $p(\alpha)$ from the opinion model perspective. Although equation (3) is still symmetric around zero and scale dependent in the tails, it presents different properties to the Gaussian and match the features we want to analyse: it is able to quantify the sole existence of plurality in the set of individuals if we assume $\mu = 0$ and $\sigma \neq 0$; this means that in assuming $\alpha = 0$ as the expected trait of the individuals, we simply have α stretching around its average behaviour. On the other limiting situation, $\mu \neq 0$ and $\sigma = 0$, we have a polarised system composed of extremely aligned partisan grouplets. Last, we have the plausible scenario of a combination of both for $\mu \neq 0$ and $\sigma \neq 0$, representing two heterogeneous grouplets that vie or rival one another [20]. In the last class, the condition $\mu = \sigma$ defines the point at which the concavity of $p(\alpha = 0)$ changes, or in other words, the existence of partisanship becomes largely screened by plurality. The reader is addressed to appendix in which tentative social experiments on some cases that fit this approach are introduced.

On the right hand side of equation (1), the second term intends to model the interaction of agent i with other agents in the community. Unlike the model introduced

[3] A risk averse individual is unlikely to become bold.

J. Stat. Mech. (2016) 063201

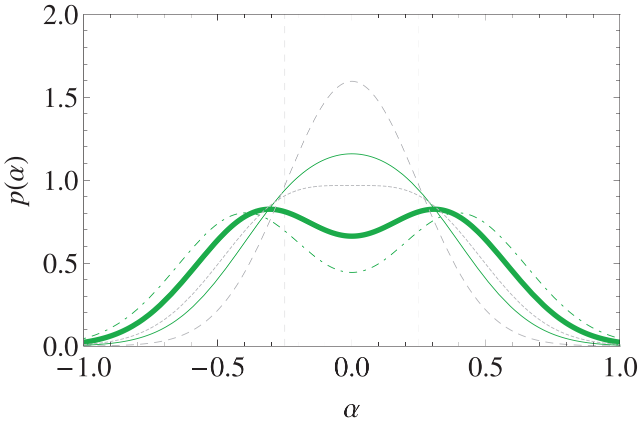

Figure 1. Probability density function of the trait parameter $p(\alpha)$ versus α. The curves go as follows: the dashed grey lines correspond to limit case distributions the bimodal distribution (extreme partisanship) with peaks at $\mu \pm 1/4$ and the Gaussian distribution (plurality and no partishanship) with $(\mu = 0, \sigma = 1/4)$. The short dotted grey line represents the 'half-way' distribution with $(\mu = \sigma = 1/4)$. For the green lines we have: $(\mu = 1/5, \sigma = 1/4)$ (full thin), $(\mu = \mu_t = 0.33\,154, \sigma = 1/4)$ (full thick), $(\mu = 2/5, \sigma = 1/4)$ (dot–dashed). The thick line case corresponds to a special case in our model, as shown in section 3.

by Deffuant and collaborators [11] and some of its generalisations [12], we let all the individuals influence and be influenced irrespective of the difference of stances s_i and s_j. The analytical form of this interaction is given by

$$U_i^{\rm int}(\mathbf{s}) = \frac{k}{4N} \sum\nolimits_j (s_i - s_j)^2, \tag{4}$$

where the factor N^{-1} is introduced to guarantee the extensivity of $U^{\rm int} \equiv \sum_i U_i^{\rm int}$; i.e. $U^{\rm int}/N =$ const when $N \to \infty$. It is important to notice that the independence on the difference between stances is relevant when the group of individuals must reach a settlement and therefore the decision of only interacting with only people who think alike is believable to be ineffective.

That said, the dynamics of the stance of agent i, s_i, is given by[4]

$$-\frac{\mathrm{d}s_i}{\mathrm{d}t} = a\, s_i + b\, s_i^3 + (s_i - S) - \alpha_i, \tag{5}$$

with $S \equiv N^{-1} \sum_i s_i$ representing the overall macroscopic state of the ensemble of individuals, i.e. the *collective stance*. Herein, we define that for $S \neq 0$ the individuals converge to a stance that is different to that we assumed as the expected outcome; hence, we christen it *nontrivial* collective stance. Complementary, the collective stance $S = 0$ represents the *trivial* collective stance that corresponds to the most likely outcome for the subject under the decision-making process[5].

[4] Hereinafter, parameters a, b and $\{\alpha_i\}$ have been rescaled to k/N units so that the previous coupling parameter becomes $k/(4\,N) = 1$.

[5] Retrieving the example of the investment in stocks, $S = 0$ can be established by shifting the values of the individual stances, s, by the previously referred standard value of 10% of the wealth allocated to risky investments.

J. Stat. Mech. (2016) 063201

From equation (5), we understand that its individual cost function U corresponds to,

$$U_i = \frac{a+1}{2} s_i^2 + \frac{b}{4} s_i^4 - (S+\alpha_i) s_i + \frac{S^2}{2}. \tag{6}$$

Equation (6) thus represents the competition between accompanying the collective stance S—described under the term $S^2/2$—and going wayward to assume another stance that stems from her personal vision on the subject which is quantitatively represented by her value of α (the impact of the explicit form of U_i will be further discussed in section 4).

Individually, the set off between the two factors corresponds to achieving an individual optimal stance, s_i^*, given by the condition

$$\left.\frac{\mathrm{d}s_i}{\mathrm{d}t}\right|_{s_i=s_i^*} = 0 \tag{7}$$

or equivalently,

$$(a+1)\, s_i^* + b\, {s_i^*}^3 - S - \alpha_i = 0, \tag{8}$$

where s_i^* is both a function of α and the collective stance, $S = S(\mathbf{s}^*)$ notoriously constituting a self-recursive problem. Owing to the fact that we have considered that the individuals interact globally, for a sufficiently large set the problem is solved by means of considering[6],

$$S = \int s^*\, p(\alpha)\, \mathrm{d}\alpha. \tag{9}$$

According to the theory of cubic functions, the number of real solutions to equation (8) is given by the sign of the discriminant,

$$\Delta = 4b(a+1)^3 + 27b^2(S+\alpha)^2. \tag{10}$$

When $\Delta < 0$, we have three real solutions; for $\Delta > 0$, the equation provides just a real zero, whereas for vanishing Δ multiple roots exist. Again, we can understand that the value of a is crucial for the number of existing roots concurring with the change in the concavity of U_i at $s_i = 0$. Specifically, for $a < -1$, $U_i(s_i = 0)$ is concave and convex otherwise.

In furtherance of simplicity, we focus on the stable solutions to systems with $b = 1$ and $-1 < a < 0$, which already exhibit a rich behaviour, as we will show[7]. In the case $\Delta > 0$, the real solution to equation (8) reads $s^* = y - (a+1)/(3y)$ where,

$$y = \sqrt[3]{\frac{S+\alpha}{2} + \sqrt{\frac{(S+\alpha)^2}{4} + \frac{(a+1)^3}{27}}}. \tag{11}$$

In the limit $N \to \infty$, S is evaluated self-consistently and it can be split into two terms,

[6] Under other conditions like assuming a complex network topology or a Deffuant *el al* interaction this way of solving the problem would represent a mean-field solution.

[7] It must be noted that by assuming $b = 1$, we are (once again) merely normalising the other parameters without affecting the generality of the results.

$$S = \frac{1}{2}\sum\nolimits_{\pm}\int s^{*}(S \pm \mu + \zeta\,\sigma)\,g(\zeta)\,\mathrm{d}\zeta, \tag{12}$$

$$= \frac{1}{2}\sum\nolimits_{\pm}\int s^{*}(\Psi^{\pm})\,g(\zeta)\,\mathrm{d}\zeta = S^{+} + S^{-}, \tag{13}$$

where $g(z)$ is the Normal distribution and

$$\Psi^{\pm} \equiv S \pm \mu + \zeta\,\sigma. \tag{14}$$

From equation (13), the overall solution corresponds to the average of left-hand and right-hand solutions; that recovers the priorly mentioned 'tug-of-war' picture of two vying blocs where $S^{\pm}$ can be viewed as the stance assumed by the each of them. The above integrals are then expanded reading,

$$S^{\pm} = \frac{1}{2}\int\left[s^{*}(\Psi_{0}^{\pm}) + \sum_{n\geqslant 1}\frac{1}{n!}\left.\frac{\partial^{n}s^{*}}{\partial\,(\Psi^{\pm})^{n}}\right|_{\Psi_{0}^{\pm}} s^{n}\right]g(\zeta)\mathrm{d}\zeta. \tag{15}$$

Changing ζ for $-\zeta$ in the Ψ_{0}^{-} terms and taking into consideration the antisymmetrical property of the solutions to equation (8), every even term of the expansion in S vanishes yielding

$$S = c_1\,S + c_3\,S^{3} + c_5\,S^{5} + \mathcal{O}(S^{7}). \tag{16}$$

Equation (16) is analogous to the critical equations where their solutions are defined by the intersection of a sigmoid-like function and the bisection of the odd-quadrants. It is thus expected that for $c_1 > 1$ we have a collective stance characterised by $S \neq 0$. It is thus worthwhile to analyse in what conditions—namely the values of the parameters a, μ and σ—we can observe a such outcome.

3. Results

In order to obtain the line separating off the region $S = 0$ from its counter-region $S \neq 0$, we need to compute the solutions to the condition

$$c_1 = \int\mathcal{A}\frac{a+1+3\,\mathcal{B}}{9\,\mathcal{B}^{2}}g(\zeta)\,\mathrm{d}\zeta = 1, \tag{17}$$

with,

$$\mathcal{A} = \frac{1}{2} + \frac{\mu+\zeta\,\sigma}{4\,\mathcal{C}},\quad \mathcal{B} = \left[\frac{1}{2}(\mu+\zeta\,\sigma) + \mathcal{C}\right]^{\frac{2}{3}},\quad \mathcal{C} = \sqrt{\left(\frac{a+1}{3}\right)^{3} + \left(\frac{\mu+\zeta\,\sigma}{2}\right)^{2}}. \tag{18}$$

The left panel in figure 2 shows the solutions to equation (17) for the case of pure plurality, $\mu = 0$. In this case, as well as in any other situations for which $c_3 < 0$, we pass from $S \neq 0$ to $S = 0$ (and vice-versa) in a continuous way. However, that inequality is not always observed for c_3; that being the case and $c_3 > 0$, we have a discontinuous

J. Stat. Mech. (2016) 063201

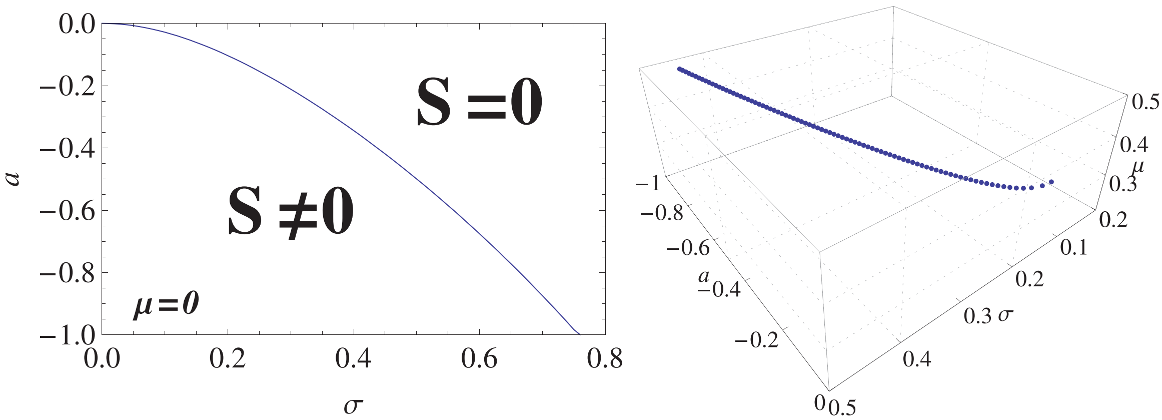

Figure 2. Left panel: phase diagram in the a versus σ plane for situations exhibiting only plurality, $\mu = 0$. In this case, the line separating off the region of trivial collective stance $S = 0$ from nontrivial collective stance, $S \neq 0$, corresponds to a continuous transition. Right panel: tricritical frontier for values of a between 0 and -1. The limiting values are made explicit in the text.

change of the collective stance when comparing systems with parameters infinitesimally close but on each side of the critical line.

That been said, we note the line separating communities that produce trivial and nontrivial collective stances can be composed of segments describing continuous and discontinuous transitions. The region where the transition changes its nature is defined as a tricritical point(line). In the right panel of figure 2, we have the tricritical line given by the conditions $c_1 = 1$ and $c_3 = 0$ to which we will get back.

For perfectly polarised systems, $\sigma = 0$, the tricritical point is located at $a_t = -1/6$ and $\mu_t = 0.209\,513\ldots$ and within the region of stability of U_i, $a > -1$. The tricritical line finishes at $\sigma \approx 0.441\,12(8)$ and $\mu \approx 0.474\,90\ldots$.

In the case of exclusively plural systems, $\mu = 0$, the problem is akin to the random field Ising model (with a continuous symmetry group (SG) instead of the Z_2 SG of Ising models though) for which only continuous transitions take place [24].

For systems with partisanship, $\mu \neq 0$, figure 3 shows the frontier between trivial and nontrivial domains can be composed of continuous and discontinuous transition regions as we have said. For the green frontier ($\sigma = 1/4$), the shape of $p(\alpha)$ for $\mu = 1/5$ (continuous), $\mu = \mu_t = 0.331\,54$ (tricritical) and $\mu = 2/5$ (discontinuous, as computed shortly) is shown in figure 1.

As any system that is described within a probabilistic context, the optimal collective stance must be such that given $\{a, \mu, \sigma\}$ the value of S optimises the entropy of the system, Σ. For our problem, we interpret the entropy as a measure of the heterogeneity so that $\Sigma = 0$ implicates that all the agents have the same optimal stance $s_i = S$.

Hamiltonian equilibrium systems are also treated in probabilistic terms using statistical mechanics and thermodynamics. In that case, a discontinuous (first-order) transition is defined by having a jump in the entropy of the system with both phases being thermodynamically equivalent and obtained by means of the so-called Maxwell construction, which leads to the existence of a latent heat [22].

Taking into consideration the critical features we have observed for our out-of-equilibrium case, we translate the thermodynamical formalism into the decision-making

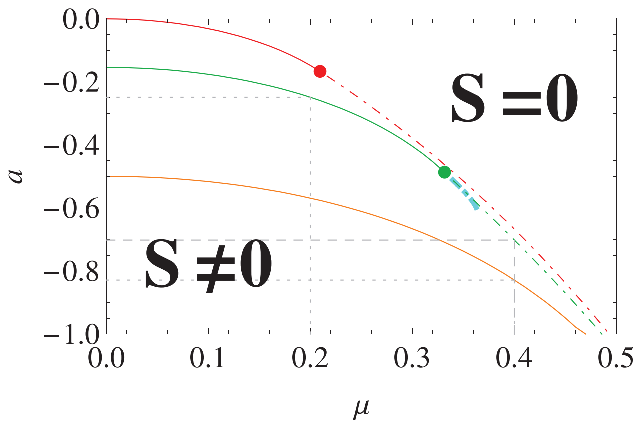

Figure 3. Phase diagram $S \neq 0 \leftrightarrow S = 0$ (from top to bottom: $\sigma = 0, 1/4, 1/2$). The full (dot–dashed) lines indicate a continuous (discontinuous) and the dots the tri-critical points located at ($\mu = 0.209\,513\ldots, a = -1/6$) for $\sigma = 0$ and ($\mu = 0.331\,54, a = -0.486\,58$) for $\sigma = 1/4$. Since $\sigma = 1/2 > 0.441\,129$, there is no discontinuous transition for $a > -1$. The thick dot–dashed turquoise line depicts the discontinuous transition line around the tricritical point obtained from equation (22). The horizontal and vertical lines indicate the critical points of figure 4 whose distribution of the trait parameter, $p(\alpha)$ is depicted in figure 1.

process framework in the same way thermodynamical entropy can be put into information entropy: first, we assert that the existence of a discontinuous transition means that, in respect of its cost, the individuals cannot collectively establish a difference between assuming the trivial and a nontrivial stance. Analytically this reads,

$$\mathcal{U}(S \neq 0) = \mathcal{U}(S = 0) \tag{19}$$

with,

$$\mathcal{U}(S) \equiv \int U(\alpha, S)\, p(\alpha)\, \mathrm{d}\alpha, \tag{20}$$

where U and S are respectively defined by equations (6) and (13); second, making use of the intimate relation between (the change of) entropy and heat for equilibrium systems, we learn that a discontinuous change between a nontrivial collective stance and $S = 0$ implies the existence of a latent heterogeneity in the system, latent in the sense that it does not influence the value of the cost function to make the system sway towards the trivial collective stance.

In figure 4, we show the transitions at the points indicated in figure 3, with $|S|$ obtained from the computation of equations (5) and (6). It is plain that for $\mu = 2/5$ and $\sigma = 1/2$, the transition is discontinuous whereas for the other instances we have plotted the transition is continuous.

The existence of discontinuous transitions is associated with the appearance of metastability in the value of S, which otherwise could only be obtained letting the system assume values of $|a|$ larger than 1. Moreover, metastability points out that the initial condition can be relevant for the long-term collective stance of the system. The analysis of the regions giving rise to metastable states are best obtained using the dual approach presented nextly.

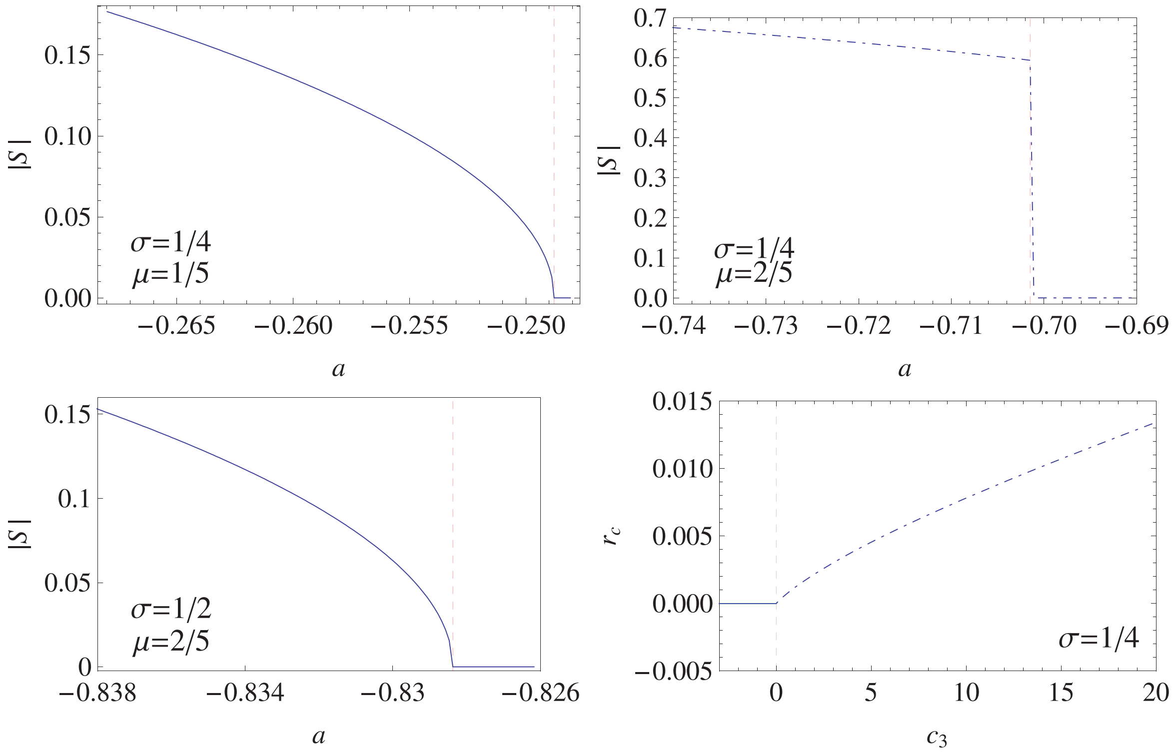

Figure 4. Upper panels: $|S|$ versus a for ($\sigma = 1/4, \mu = 1/5$) (left) and ($\sigma = 1/4, \mu = 2/5$) (right). The transitions at $a_c = -0.248\,79$ and $a_c = -0.701\,50$ are continuous and discontinuous, respectively. Lower panels: $|S|$ versus a for ($\sigma = 1/2, \mu = 2/5$) (left) with a continuous transition coming about at $a_c = -0.828\,37$; the *lambda* continuous critical line $r_c = 0$ for $c_3 < 0$ and the discontinuous critical line for $c_3 > 0$ for $\sigma = 1/4$ (right).

The current problem can be alternatively analysed looking at its optimal dual [21], $\mathcal{D}$ in the same way that instead of analysing the internal energy of an equilibrium system we can best study a problem using the thermodynamical potentials [22]. In that case, we intuitively understand that $\mathcal{D}$ tries to represent the superposition of two effects: the minimisation of U and the optimisation of $\boldsymbol{\Sigma}$. That means that under some conditions, the entropy (heterogeneity) overcomes U and the individuals end up assuming the trivial collective stance whereas for other groups of people with different parameter things go the other way round and we get a nontrivial stance. Since the optimal solution has to be the same independent of the way we tackle the problem, the equation for the extrema of $\mathcal{D}$,

$$\frac{\mathrm{d}\,\mathcal{D}}{\mathrm{d}\,S} = 0 \tag{21}$$

must concur with with equation (16) which imposes

$$\mathcal{D} = \frac{1 - c_1}{2}\,S^2 - \frac{c_3}{4}S^4 - \frac{c_5}{6}\,S^6 + \mathcal{O}(S^8), \tag{22}$$

[$\mathcal{D}(0) = 0$] in the vicinity of the transition. As we move away from that region, the difference between $\mathcal{D}$ and $\mathcal{U}$ increases due to the fact that the jump in S of the discontinuous transition soars and for this reason we only compare in figure 3 the results given by both approaches in the neighbourhood of the tricritical point.

As previously computed, the coefficient c_1 assumes any sort of real value depending on the variance of the trait parameter, α, σ_α^2. That dispersion originates in the levels of plurality, σ and polarisation, μ—actually μ^2—presented by the horde. The same occurs for c_3 and c_5, whose values are provided by equations (13)–(16).

For the sake of simplicity, we keep the level of plurality, σ, constant and focus on the dependence on the degree of partisanship, μ. Moreover, inasmuch as we are looking for stable solutions, we must bear in mind that when c_3 is positive, the sixth order term must be guaranteed positive and and it can be neglected otherwise. In that case, it is easily verifiable that as the value of $c_1(\sigma_\alpha)$ increases(decreases), symmetric minima of $\mathcal{D}$ loom corresponding to the emergence of metastable nontrivial collective stances. These are classified as metastable because they are characterised by a value of S that is just a local minimum; the most stable solution is still the trivial stance $S = 0$. That signifies that its measurement depends on the initial conditions the individuals have; in other words, if we start close to a metastable nontrivial collective stance, $S \neq 0$ can endure in the long term, but dissimilar initial states will still evolve to $S = 0$. Thus, even if the heterogeneity of the trait hints that the final collective stance is most surely the trivial one, the starting point for reaching a collective position is such that they get trapped in a good but not the best option, colloquially the (in)famous 'this is the best we could get' outcome.

The existence of a set of values for which we have metastable stances is upper bounded by the solution to the double condition,

$$\left.\frac{\partial\,\mathcal{U}}{\partial\,S}\right|_{S\neq 0} = \left.\frac{\partial^2\,\mathcal{U}}{\partial\,S^2}\right|_{S\neq 0} = 0, \qquad (23)$$

which we define as the *stability limit on heterogeneity*, because it corresponds to the maximum scenario we could obtain $S \neq 0$ by increasing σ_α.

In the opposite direction, if we keep on decreasing(increasing) $c_1(\sigma_\alpha)$ from the critical situation, $\mathcal{U}(S \neq 0) = \mathcal{U}(S = 0)$—or $\mathcal{D}(S \neq 0) = \mathcal{D}(S = 0)$—, we will reach a point at which the trivial stance ceases being a local minimum and only the symmetric nontrivial collective stances $S \neq 0$ perdure. This situation is determined by

$$\left.\frac{\partial^2\,\mathcal{U}}{\partial\,S^2}\right|_{S=0} = 0, \qquad (24)$$

and will be defined as the *stability limit on homogeneity* because it corresponds to the extreme situation for which in decreasing σ_α it is still possible to obtain $S = 0$. A representation of all these stability limits is shown in figure 5 for $\sigma = 1/10$.[8]

[8] In practice, we verify that for a fixed value of σ, there exists a value, $\mu = \mu^*$, beyond which it is not possible to accurately determine this limit because $\mathcal{D}$ gets extremely flat that at $S \lesssim 0$—where the concavity of $\mathcal{D}$ changes—are not feasibly detected performing the calculations of the second derivatives with accuracy up to 8 significant digits.

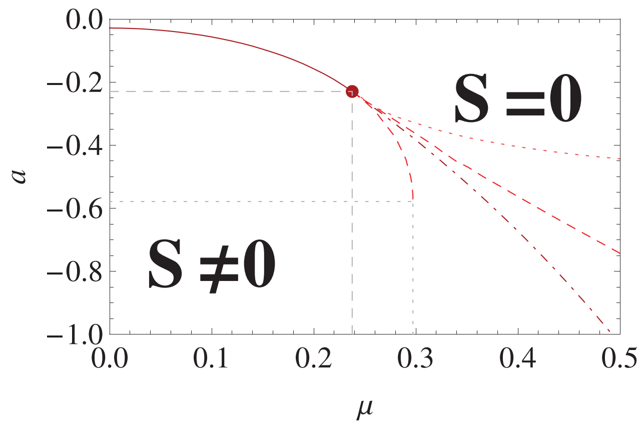

Figure 5. Extended phase diagram a–μ for $\sigma = 1/10$. The full burgundy line represents the continuous transition line, which finishes at the tricritical point located at ($\mu = 0.237\,58$, $a = -0.229\,65$). The critical frontier goes on as a discontinuous transition depicted in the form of the burgundy dot–dashed line. Above and below it, we depict the stability limits on heterogeneity and homogeneity, respectively, that are computed as indicated in the main text. We also plot in the form of a dotted red line the bound representing the change of concavity in $\mathcal{U}(S)$ with $S \neq 0$ that is the forerunner of the upper metastability critical line.

Still, we can say something more about the points of transition. Looking at equation (22), we verify that if $c_3 < 0$ the critical condition corresponds to $c_1 = 1$ whereas if $c_3 > 0$ the critical value reads $c_1 = 1 - \frac{3\,c_3^2}{16\,|c_5|}$, so that in the limit of $c_3 \to 0$ we obtain $c_1 = 1$. The parabolic relation between c_1 and c_3 is presented in a panel of figure 4 for $\sigma = 1/4$. As said, for $c_3 < 0$ the transition is continuous and hence the response of the system to variations on the heterogeneity σ_α, namely μ—that is computed from $\partial^2\mathcal{U}/\partial\sigma_\alpha^2$—evinces a divergence at the transition corresponding to an out-of-equilibrium 'lambda line' as first observed for the equilibrium superfluid Helium transition [23]. This means that close to the transition $S \neq 0 \leftrightarrow S = 0$ the fluctuations in the cost function are significant when performing averages over an ensemble of decision-making processes with critical parameters (or close to that).

For the case of $c_3 > 0$, the discontinuous nature of the transition implies a jump in the value of S that is also related to a discontinuity in the entropy, i.e. there is a given degree of heterogeneity—which is quantified by a given amount of entropy, ℓ—that does have no impact on the value of the collective stance. That entropy is the latent heterogeneity we have made mention to. Bridging $\mathcal{D}$ and Σ from Thermodynamics and dual optimisation formalism [21, 31] we obtain,

$$\ell \propto \frac{3\,c_3}{8\,|c5|}, \tag{25}$$

whose behaviour is plotted in figure 6 for the case of figure 5, $\sigma = 1/10$. Equation (25) indicates that at the tricritical line presented in figure 2 is continuous; however, if close to the transition we linearise $c_1 \propto \Delta\sigma_\alpha^2 \equiv \sigma_\alpha^2 - \hat{\sigma}_\alpha^2$, using equation (22) we have $S \propto (\Delta\sigma_\alpha^2)^{1/4}$ that contrasts with the Ising exponent of $1/2$ (the hat denotes criticality).

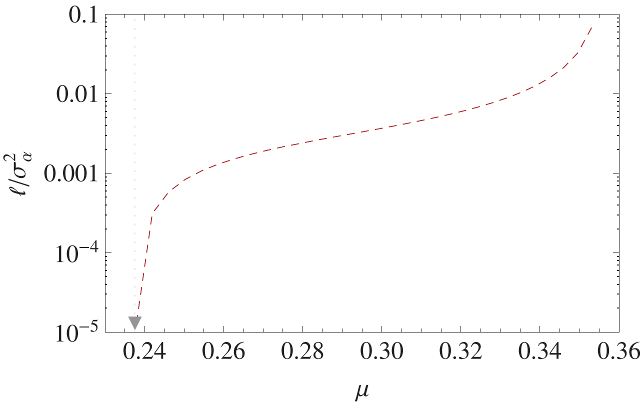

Figure 6. Latent heterogeneity (scaled by σ_α^2) versus the partisanship parameter μ for discontinuous transitions with $\sigma = 1/10$. The symbol ▼ and the dotted vertical line indicate the μ value of the tricritical point $\mu_t = 0.237\,58$ for which there is no latent heterogeneity as the transition is continuous.

4. Final remarks

In this work, we have surveyed the impact of different types of diversity in a social system, namely partisanship and plurality, have in the achieving of a collective stance when the range of allowed opinions is continuous. The stance vented by of each individual during the decision-making process is influenced by a given trait of her personality which is eminent to problem under discussion. For instance, the level of risk aversion for financial management choices, the degree of libertarianism that shape the views regarding the extent of a penalty over an offense or the left–right political location on a discussion about the upper band value for income taxes that can be reckoned fair. This trait is classified by a quantitative proxy, α. In order to capture the two kinds of diversity we have assumed, its distribution follows a double-Gaussian. In doing so, we have a distribution, $p(\alpha)$, that is able to mimic a range of situations that goes from the case where the individuals are divided into two perfect antagonistic partisan groups by making the variance of the Gaussians go to zero up to situations corresponding to the emergence of an effective single group with a given level of plurality when the distance between the peaks on each side of the distribution vanishes, $\sigma = \mu$. The distribution also contains the case of pure plurality when $\mu = 0$.

The first point it was shown is that, contrarily to systems wherein only plurality is present, societies that are also characterised by partisanship—and resulting in a combination of two antagonistic inhomogeneous blocs—can undergo both continuous and discontinuous transitions from collective states that we have defined as the trivial and nontrivial stances, $S = 0$ and $S \neq 0$, respectively. Explicitly, depending on the values of the parameters of the problem—especially σ and μ—the system will start at some initial stance, S_0, and evolve into a final trivial(nontrivial) situation. Altering the conditions of the system—e.g. changing the elements so that $p(\alpha)$ or other conditions of the problem modify—we can go from one final stance to the other with the class of that transition depending on the balance between both the plurality and the partisanship. When the plurality is sufficiently large to smear the difference between the peaks of $p(\alpha)$, the system is inclined to experience a continuous transition similarly to what happens in the case of the centred Gaussian representing pure plurality. On the other hand, when partisanship is effective, the system is likely to experience a jump in the value of S when going from trivial to nontrivial collective stances. From the jump quelling perspective, we perceive the role that plurality can have in turning transitions smoother, something that can be useful in many situations. We shall mention that discontinuous-like transitions in the collective state of a group are observed in a variety of systems: from human (social and financial) systems to animal collective behaviour [25–29].

That said, we retain that although both cases, plurality and polarisation (partisanship), correspond to indications of diversity in the system, they actually have a different impact in the long-term macroscopic stance. Specifically, the existence of polarisation in the trait of the individuals induces more abrupt changes than cases with simple plurality. Thence, the existence of partisanship in a group permits the development of metastability in a regime ($|a| < 1$) that is not possible to achieve in similar systems where only diversity is taken into account [13] and that resemble the Random Field Ising model [24] within an equilibrium physical context (letting variable s symmetry group matters out). That metastability is particularly relevant since, as we have observed, for given levels of diversity and initial conditions, the group can attain a nontrivial collective stance by changing, namely increasing, the diversity in the system. This fact bestows a counter-intuitive role of consensus ($S \neq 0$) enhancer to the diversity in continuous opinion models, an effect that can also be obtained resorting to pinpointed time dependencies in stochastic quantities that stoke resonance mechanisms [30]. Moreover, since the existence of discontinuous transitions is associated with a jump in the entropy, a system well-described by this model is able to display a property that we have defined as latent heterogeneity; latent because it is somewhat hidden and incapable of creating a cost difference between assuming the trivial and a nontrivial stance.

Concerning future work, several approaches can be carried out. Besides performing field studies (or 'social experiments' [33]) along the lines sketched in appendix, there is a set of complementary theoretical analyses and variants to the model that are worth studying. With respect to the former theoretical case, we mention further analyses on the critical behaviour of the model, namely the computation of critical exponents and its relation to finite-size scaling. As shown, along the tricritical line the collective stance changes differently to the Ising class. Assuming the universal validity of the hyperscaling relations [31] we expect that the remaining critical exponents in the Ising-unlike region are different as well. It is important to stress that critical exponents play an important role in the estimation of the critical properties of finite-size systems (small groups). The model of [24]—which is similar to this manuscript proposal, but whose critical behaviour is just continuous—allowed finding that out-of-equilibrium(nonconservative) systems can present finite-size scale exponents obtained by numerical analysis that do not match the exponents theoretically computed. It is would be thus relevant to probe whether this feature remains in Ising-unlike situations or how this model critically behaves beyond the region $|a| > 1$. Last, taking into consideration that the model introduced exhibits regions of meta-stability, a relevant subject of study concerns the network topology of the so-called Lyapunov potential $U(\mathbf{s})$, namely the network of minima generated according to earlier published work [32].

Within this context, we can briefly discuss the role of the specific form of the cost function; we reckon that as long as its main properties—namely stability and the possibility of having a change in the concavity at $s_i = 0$—are preserved, there will be no significant modifications to the qualitative behaviour of the problem. In other words, we will still be able to obtain continuous and discontinuous transitions depending on the interplay between polarisation and plurality. Nevertheless, properties like the time taken to reach a stable solution will naturally depend on the specific form of U_i. A good example is to replace the polynomial form $a\, s_i^2/2 + b\, s_i^4/4$ in the cost function by a logarithmic form; for that particular case, we expect a significant increase in the transient time for large values of the initial condition, S_0 which are distant from the stable collective stance as well. Along the same lines, if we consider a distribution that is different from the present double-Gaussian, the qualitative results will not alter unless the new $p(\alpha)$ is unable to represent systems with bipartisanship and/or plurality and vice-versa.

Still in the field of variants we can, e.g. analyse the consequences of assuming a time dependent α or multiplicative (noise) α. Another change worth considering is to modify the interaction term, namely by bearing in mind the alikeness between the individuals—approaching Deffuant *et al* model and generalisations—, their reputations or even the willingness to cooperate with the others in the coupling parameter $k \to k_i$, which will certainly provide even reacher behavioural diagrams.

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Appendix. Outline of an experimental approach for checking the model

In the current appendix, it is presented a sketch of how the behaviour described in this model can be empirically probed in a financial scenario by defining a test with panelists. The test consists of a game in which an investment club is to be implemented and its members must decide the stake of the group's portfolio that is going to be allocated to risky investments—namely, stocks—with the remaining deposited in a risk-free savings account.

- The first stage of the study concerns the estimation of the parameters. To that we consider a preliminary test based on prospect theory [34]. Each individual is asked whether she would rather receive a token of X (units) without doing anything else or enter a head or tails game where she wins Y (units) if she bets on the right outcome and 0 otherwise. A purely rational choice implies that, as long as $Y > 2\,X$, it makes statistical sense to pick the coin tossing option. However, people seldom use rationality in these cases and their options are guided by its

J. Stat. Mech. (2016) 063201

risk aversion/propensity ethos. That trait can be quantified resorting to escort distributions [35], where the actual distribution—in this case 1/2 for both winning and losing assuming a fair coin—is modified by a parameter θ_i for each individual i. The parameter is then obtained from

$$\theta_i = \ln 2\, X - \ln\, Y^*_i. \tag{A.1}$$

when i accepts the heads or tails option considering the threshold prize Y^*.

- With the set of values $\{\theta\}$ in hand, we obtain the parameters of the model by assuming them as convenient functions of θ. Explicitly, by taking into account that we want to strictly relate the heterogeneity in the system to the risk trait, panelists should be divided by their age, gender, education, whence we evaluate the average level of risk aversion for the respective group. This average value is then associated with parameter a. Afterwards, within every coherent group (defined by a) different distributions of α can be obtained using standard techniques of pooling so that it is possible to obtain a cast of individuals that abide by the conditions $\{a, \sigma, \mu\}$ we aim at.
- After selecting the individuals according to the situation we want to assess, a stance regarding the percentage of the wealth of the group that should be invested in stocks is handed to each person and the decision-making process begins. At every time, the individuals have access to the stances of all the others. To reflect global coupling of the theoretical proposal the stances must be disclosed anonymously. The reference value, $s_i = 0$, that is associated with the trivial collective stance, $S = 0$ can be taken from the 10% wedge, traditionally used in the industry.

Equivalent approaches can be set up for problems with different nature, e.g. economical (upper bound for income tax) or social (the penalty for a given wrongdoing). In those cases, we can use well-known tests like 'the political compass' [36] to classify the individuals in respect of the economical or social traits that are relevant for each problem.

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