Noise, synchrony, and correlations at the edge of chaos

Alessandro Pluchino,[1,2] Andrea Rapisarda,[1,2] and Constantino Tsallis[2,3]
[1] *Dipartimento di Fisica e Astronomia, Università di Catania and Istituto Nazionale di Fisica Nucleare sezione di Catania, Via S. Sofia 64, 95123 Catania, Italy*
[2] *Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems 22290-180, Rio de Janeiro, Rio de Janeiro, Brazil*
[3] *Santa Fe Institute, Santa Fe, New Mexico 87501, USA*
(Received 13 June 2012; revised manuscript received 23 January 2013; published 19 February 2013)

We study the effect of a weak random additive noise in a linear chain of N locally coupled logistic maps at the edge of chaos. Maps tend to synchronize for a strong enough coupling, but if a weak noise is added, very intermittent fluctuations in the returns time series are observed. This intermittency tends to disappear when noise is increased. Considering the probability distribution functions (pdfs) of the returns, we observe the emergence of fat tails which can be satisfactorily reproduced by q-Gaussians' curves typical of nonextensive statistical mechanics. The interoccurrence times of these extreme events are also studied in detail. Similarities with the recent analysis of financial data are also discussed.

DOI: 10.1103/PhysRevE.87.022910 PACS number(s): 05.45.Ra, 74.40.De, 87.19.lm

Since the nonlinear phenomenon of synchronization was first observed and discussed in the 17th century by Huygens, it has become of fundamental importance in various fields of science and engineering. It is frequently observed in complex systems such as biological ones, or single cells, physiological systems, organisms, and even populations. Synchrony among coupled units has been extensively studied in the past decades, providing important insights on the mechanisms that generate such an emergent collective behavior [1–5]. In this context coupled maps have often been used as a theoretical model [6]. Actually, many biological complex systems operate in a noisy environment and most likely at the edge of chaos [7,8]. Therefore studying the effect of a weak noise in this kind of coupled systems could be relevant to understand the way in which interacting units behave in real complex systems like, for example, living cells [9,10]. Generally speaking, random noise is considered a disturbance, i.e., something to avoid to obtain precise measurements or to minimize numerical errors. But, if on one hand it is very difficult to completely eliminate the effect of noise, on the other hand it can frequently have even a beneficial role. Among the many examples in physics and biology we may cite stochastic resonance [11], noise enhanced stability [12], induced second-order-like phase transitions [13], or enhanced diffusion in communication networks [14]. Recently random strategies have been demonstrated to be very successful also in minority and Parrondo games [15,16] and in sociophysics models related to efficiency in hierarchical organizations [17,18] or even in Parliament models [19].

In previous studies [20,21] the effect of a small noise on globally coupled chaotic units was presented for several kinds of systems and a universal behavior related to the Lyapunov spectrum was found to be a common feature. Power-law correlations and intermittent behavior have also been observed in lattices of logistic maps when some kind of global coupling exists among them, see, for example, Refs. [22,23]. In this paper, we consider a linear chain of N locally coupled logistic maps and we explore the role that a small random noise can have in creating a strong intermittent behavior and its influence on the synchronization patterns. We consider only local coupling and long-range correlations induced by the noisy environment in which our maps are embedded. At variance with previous studies, maps are not in a chaotic regime, but at the edge of chaos, where the Lyapunov exponent is vanishing [24,25]. Moreover, to investigate our intermittent behavior, we study the probability distribution functions (pdfs) of the returns of our fluctuating time series, as successfully done in different contexts for models showing self-organized criticality [26–28]. Our results can also be framed in the context of nonextensive statistical mechanics [29–31] and analogies with recent findings for stock market data analysis [32,33] will be addressed.

The model of a linear chain of N coupled logistic maps is the following

$$x_{t+1}^{i} = (1-\epsilon)\, f\left({x_t}^{i}\right) + \frac{\epsilon}{2}\left[f\left({x_t}^{i-1}\right) + f\left({x_t}^{i+1}\right)\right] + \sigma(t), \tag{1}$$

where $\epsilon \in [0,1]$ is the strength of the local coupling of each map with its first neighbors sites on the chain and the additive noise $\sigma(t)$ is a random variable, fluctuating in time but equal for all the maps, uniformly extracted in the range $[0,\sigma_{\max}]$. In our case the ith logistic map at time t is in the form $f(x_t^i) = 1 - \mu(x_t^i)^2$, with $\mu \in [0,2]$ and with $f(x_t^i)$ taken in module 1 with sign (to fold the maps' outputs back into the $[-1,1]$ interval when the noise takes them out of it). The system has periodic boundary conditions. See Fig. 1 for a pictorial view.

In the absence of noise, this model was extensively studied by Kaneko *et al.* [6], in particular in the chaotic regime, where the coupled maps show different patterns of synchronization as a function of the coupling strength ϵ. Here we consider the effect of a variable addition of noise on the same system, but at the edge of chaos, i.e., at $\mu_c = 1.4011551\ldots$. Following a procedure adopted in Ref. [23] to subtract the synchronized component and keep the desynchronized part of each map we consider, at every time step, the difference between the average $\langle x_t^i \rangle$ and the single map value x_t^i.

Then we further consider the average of the absolute values of these differences over the whole system, i.e., $d_t = \frac{1}{N}\Sigma_{i=1}^{N}|x_t^i - \langle x_t^i\rangle|$ to measure the distance from the synchronization regime at time t. If all maps are trapped in some synchronized pattern then this quantity remains close

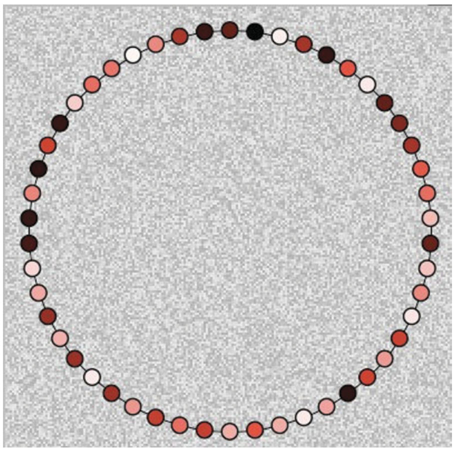

FIG. 1. (Color online) A pictorial view of a chain of $N = 50$ locally coupled logistic maps embedded in a noisy environment. The different colors indicate, at a fixed time t, different values of the maps in the interval $[-1,1]$.

to zero, otherwise oscillations are found. As commonly used in turbulence or in finance [29,32,33], we analyze these oscillations by considering the two-time returns Δd_t with an interval of τ time steps, defined as $\Delta d_t = d_{t+\tau} - d_t$.

In Fig. 2 we show that this quantity is very sensitive to the noise intensity. In Figs. 2(a) and 2(b) we plot the time evolution of Δd_t (normalized to the standard deviation of the overall sequence) for two different simulations obtained with a linear chain of $N = 100$ maps, with $\epsilon = 0.8$, $\tau = 32$ and considering the maps at the edge of chaos. For both the simulations we consider a transient of 15 000 iterations, during which the system evolves in the absence of noise ($\sigma_{\max} = 0$), then we suddenly increase the level of noise bringing it on at $\sigma_{\max} = 0.002$ [Fig. 2(a)] and $\sigma_{\max} = 0.2$ [Fig. 2(b)], respectively: It clearly appears that only in the presence of weak noise [Fig. 2(a)] the returns' time series shows large deviations from the synchronized pattern of the transient, while a higher noise intensity destroys the intermittency and induces Gaussian fluctuations.

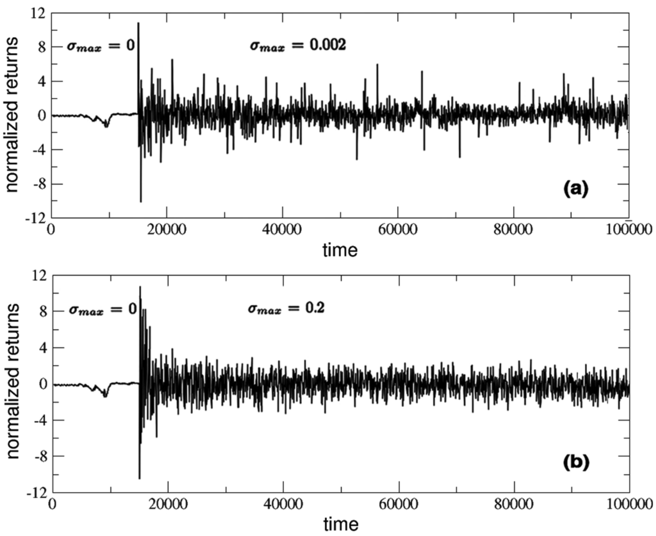

FIG. 2. We show the effect of noise in the normalized returns of Eq. (2) for the case $N = 100$, $\mu = \mu_c = 1.4011551\ldots$, $\epsilon = 0.8$, and $\tau = 32$ time steps. At time $t = 15\,000$ we switch on the noise, with $\sigma_{\max} = 0.002$ in panel (a) and $\sigma_{\max} = 0.2$ in panel (b), then we follow the maps for 100 000 iterations. See text for further details.

To quantify such a different noise-dependent behavior we plot in Fig. 3 the asymptotic pdf of normalized returns for increasing values of $\sigma_{\max}$, from 0.002 to 0.3. The fat tails in the pdfs are visible only for $\sigma_{\max} < 0.1$ and, following what was already done for a single logistic map at the edge of chaos [24,25], we tried to reproduce them by q-Gaussian curves, typical of nonextensive statistical mechanics [29] and usually found in complex systems presenting various kinds of correlations. q-Gaussians are defined as $G_q(x) = A[1 - (1 - q)\beta x^2]^{\frac{1}{1-q}}$, where q is the entropic index (which evaluates deviations from Gaussian behavior), $1/\beta$ plays the role of a variance, and A is a normalization parameter. These curves actually fit very well the numerical pdfs, as also reported in the panels of Fig. 3 (full lines). For $\sigma_{\max} = 0.002$ [Fig. 3(a)], where the tails are very pronounced, one has $q \sim 1.5$ while, for higher values of noise, the tails tend to disappear and the value of q decreases asymptotically towards $q = 1$, which corresponds to a Gaussian pdf [Fig. 3(d), with $\sigma_{\max} = 0.3$]. This definitively demonstrates that if some noise creates intermittency and correlations, too much noise destroys them.

As a further test to verify the accuracy of the q-Gaussian fit shown in Fig. 3(a), in the top panel of Fig. 4 we plot (as open circles) the q logarithm (defined as $\ln_q z \equiv [z^{1-q} - 1]/[1 - q]$, with $\ln_1 z = \ln z$) of the corresponding pdf, normalized to its peak, as a function of x^2, and we verify that a q-logarithm curve with $q = 1.54$ fits very well the simulation points with a correlation coefficient equal to 0.9958. On the other hand, in the bottom panel of Fig. 4, we show that the Gaussian behavior of Fig. 3(d) can be also obtained considering the same parameters of Fig. 3(a), but with the maps in the fully chaotic regime, i.e., with $\mu = 2$ instead of $\mu = \mu_c$. This indicates that the edge of chaos condition is strictly necessary for the emergence of intermittency and strong correlations in the presence of a small level of noise. Of course the cases $\mu = \mu_c$ and $\mu = 2$ are two limiting ones. We also checked that, changing the order parameter in the interval $[\mu_c, 2]$, other kinds of non-Gaussian pdfs occur, which are very often asymmetric or a superposition of these two extreme cases. Finally, to test the robustness of the previously found fat tails, we show in Fig. 5 the same case reported in Fig. 3(a) but now obtained with more statistics, i.e., considering 10^6 iterations and pdfs calculated with a different number of bins. The same q-Gaussian curve of Fig. 3(a) (full line) continues to reproduce very well the new data both in the central part and in the tails.

If one considers the value of the entropic index q, emerging through q-Gaussian fits of the returns' pdfs, as a measure of the correlations induced by the noisy environment on our chain of coupled maps at the edge of chaos, it is worthwhile to explore how this value changes as a function, not only of the noise $\sigma_{\max}$, but also of the number N of maps, the coupling strength ϵ, and the returns' time interval τ. We show in Fig. 6 a summary of the results obtained in this direction for $\mu = \mu_c = 1.4011551\ldots$ and changing the parameters $\sigma_{\max} = 0.002$, $N = 100$, $\epsilon = 0.8$, and $\tau = 32$ one at a time and then calculating the corresponding values of q as reported. More precisely, in Fig. 6(a) we plot the entropic index as

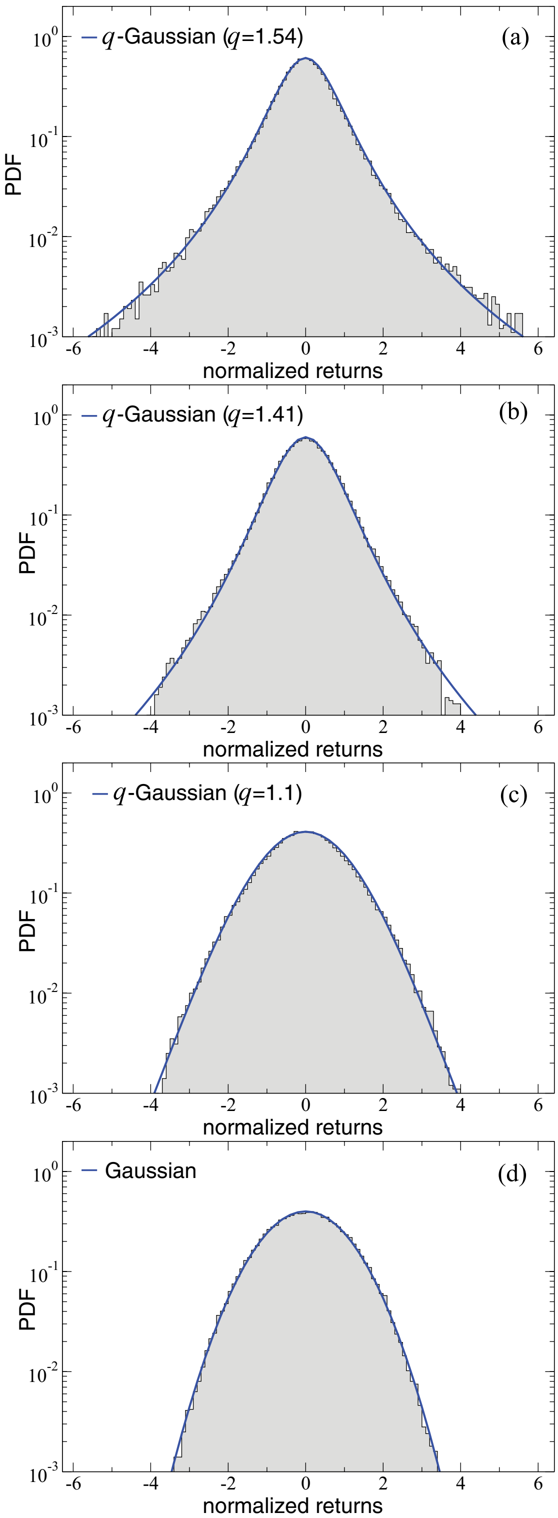

FIG. 3. (Color online) Asymptotic pdfs of the returns for $N = 100$ maps at the edge of chaos, i.e., $\mu_c = 1.4011551\ldots$, with $\epsilon = 0.8$, $\tau = 32$ (after 100 000 iterations) and for increasing values of the noise: (a) $\sigma_{\max} = 0.002$, (b) $\sigma_{\max} = 0.01$, (c) $\sigma_{\max} = 0.05$, and (d) $\sigma_{\max} = 0.3$. Fat tails are more evident for weak noise and tend to diminish by increasing noise. We also report fits of the simulation data (full curve) by means of q-Gaussian curves with values $q = 1.54$, $q = 1.41$, $q = 1.1$, and $q = 1$ (corresponding to a Gaussian), respectively. Returns are also normalized to the standard deviation to have a pdf with unit variance. See text for further details.

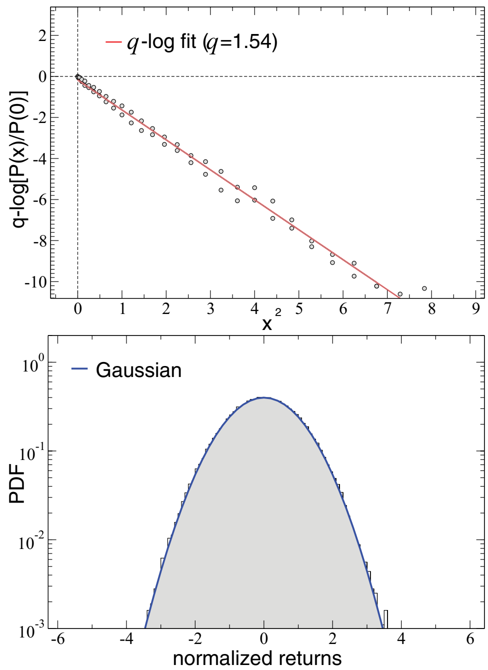

FIG. 4. (Color online) Top panel: The q logarithm of the pdf reported in Fig. 3(a) (normalized to the peak) is plotted as a function of x^2. A q-logarithmic curve with $q = 1.54$ fits the points with a correlation coefficient equal to 0.9958. Bottom panel: Same simulation reported in Fig. 3(a), but with the maps in the fully chaotic regime ($\mu = 2$). See text for further details.

function of $1/N$ and we see that q remains greater than 1 also for very large N, thus implying that the noise-induced correlations are not a finite-size effect. The influence of noise on the value of q used to fit the pdf of the returns is reported in Fig. 6(b), where an asymptotic convergence towards 1, for strong noise, and towards $\sim$1.5, for weak noise, is clearly visible. Quite interestingly, the plot of q as a function of the coupling strength, Fig. 6(c), has a maximum in correspondence of $\epsilon \sim 0.8$, a value which evidently allows an optimal spreading of correlations over the maps' chain in the

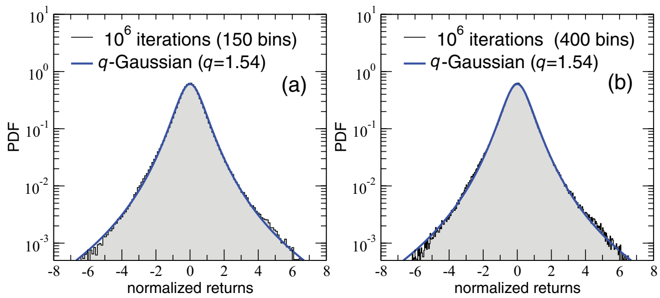

FIG. 5. (Color online) Asymptotic pdfs of the returns for $N = 100$ maps at the edge of chaos, i.e., $\mu_c = 1.4011551\ldots$, with $\epsilon = 0.8$, $\tau = 32$ as in the case of Fig. 3(a), but now taken with higher statistics to test the robustness of previous results. In this case we considered 10^6 iterations and two histograms with a different number of bins, i.e., (a) 150 and (b) 400, respectively. The same q-Gaussian curve reported in Fig. 3(a), here shown for comparison, reproduces very well the new numerical simulations independently of the size of the bins used for the histogram.

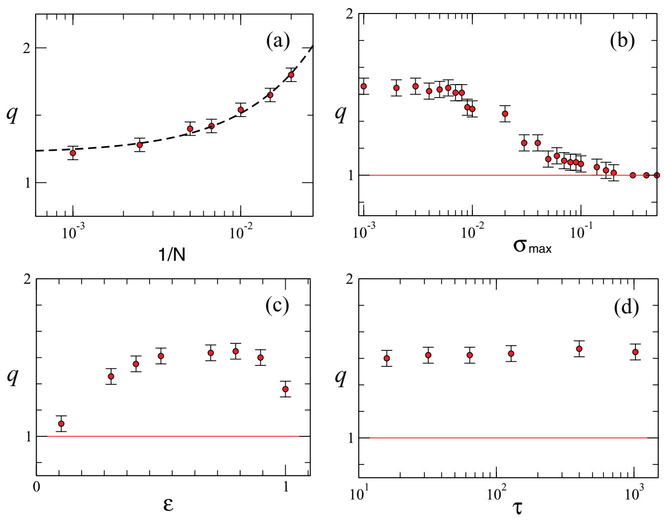

FIG. 6. (Color online) A summary of the main results found in our study as a function, respectively, of (a) the size of the system N, (b) the level of noise $\sigma_{\max}$, (c) the coupling ϵ, and (d) the returns' interval τ. The maps were always considered at the edge of chaos (where the system has a multifractal structure which is consistent with the τ independence of q). See text for further details.

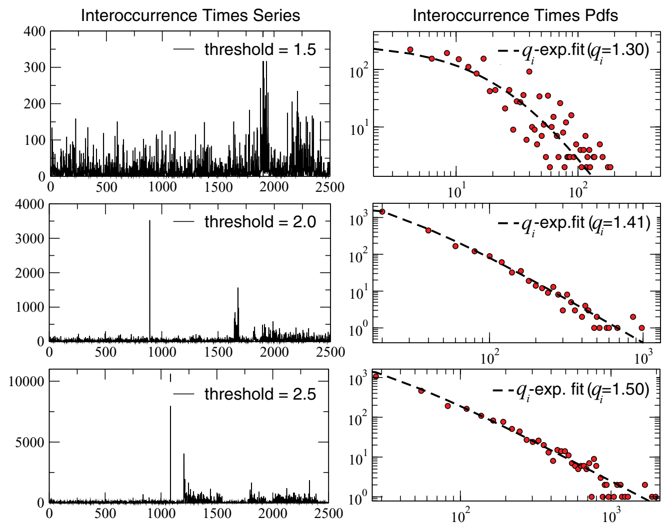

FIG. 7. (Color online) Left column panels: Plots of the interoccurrence times τ_i of returns for increasing thresholds in the case $N = 100$, $\epsilon = 0.8$, $\sigma_{\max} = 0.002$, and $\tau = 32$. Right column panels: pdf of the time series reported on the left panels. These pdfs are nicely fitted by q_i-exponential curves, whose value of q_i is also reported. See text for further details.

presence of a small noise and corresponds to the fattest tails in the pdf's, i.e., to frequent large jumps. The fact that they occur for finite levels of ϵ is somewhat reminiscent of phenomena such as stochastic resonance [11]. Finally, in Fig. 6(d) we plot q versus the time interval τ used to calculate the returns: The resulting points seem to stay constant in the range of τ explored, thus confirming the robustness of this kind of correlations for low levels of noise.

Long-term correlations in a system typically yield power-law asymptotic behaviors in various physically relevant properties. In studies of financial markets, power-law decays in the so-called "interoccurrence times" between the subsequential peaks, higher than a fixed threshold, in the fluctuating time series of returns like those shown in Fig. 2(a) were recently observed [32]. Once fixed a limit threshold, the sequence of the interoccurrence time intervals results to be well defined and it is then possible to study its pdf. We do this for our usual chain of $N = 100$ maps at the edge of chaos, with $\epsilon = 8$, $\tau = 32$, and for a weak noise with $\sigma_{\max} = 0.002$. In the left panels of Fig. 7, the interoccurrence time series for the normalized returns are plotted (from top to bottom) in correspondence of the three increasing values of the threshold (1.5, 2.0, and 2.5), while the correspondent pdfs are reported on the right. In all the cases q_i exponentials (i.e., pdf $\propto [1 - (1 - q_i)\tau_i/\tau_{q_i}]^{1/1-q_i}$, where the subindex i stands for *interoccurrence*) satisfactorily fit the data for values of q_i which depend on the threshold, in complete analogy with what was observed for the financial data [32,33].

In Fig. 8(a) we also show that q_i scales linearly as a function of the threshold. This can be considered as a further footprint of the complex emergent behavior induced on the system by the small level of noise considered. Interestingly enough, in the limit of the vanishing threshold, q_i approaches unity, i.e., the behavior becomes exponential, which is precisely what was systematically observed in the financial data [33]. Finally, we also calculated the autocorrelation function (ACF) $C_{\text{th}}(\Delta) = A' \sum_k^{L-\Delta} [\tau_i(k) - \langle\tau_i\rangle][\tau_i(k+\Delta) - \langle\tau_i\rangle]$ for the interoccurrence time series reported in the left panels of Fig. 7, where L is the length of the time series, th stands for threshold and A' is a normalization factor. As shown in Fig. 8(b), for the corresponding values of the threshold considered, we found a power-law decay $C_{\text{th}}(\Delta) \sim \Delta^{-\gamma(\text{th})}$ with the values for the exponent $\gamma(\text{th})$ decreasing with the increase of the threshold

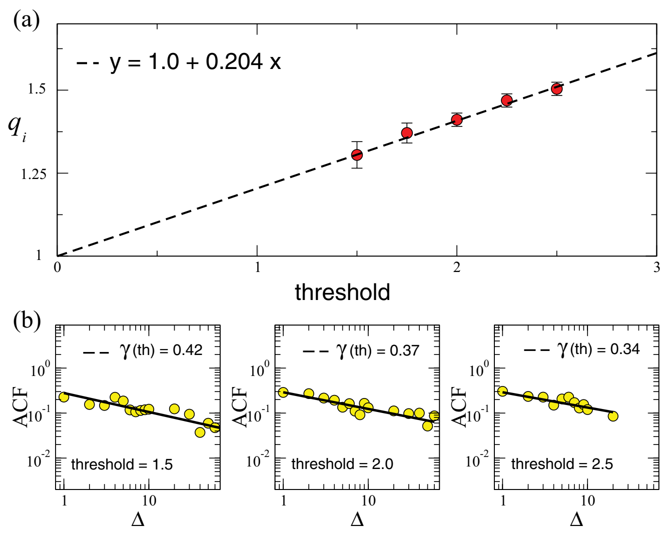

FIG. 8. (Color online) (a) The values of the index q_i, resulting by q_i-exponential fits of the interoccurrence time series pdfs, are reported in correspondence of five values of the threshold (1.5, 1.75, 2.0, 2.25, 2.5). A linear fit is also plotted for comparison. (b) The autocorrelation function $C_{\text{th}}(\Delta)$ for the interoccurrence time series is plotted for the three correspondent thresholds values of Fig. 7 and the numerical points are fitted by power-law curves $C_{\text{th}}(\Delta) \sim \Delta^{-\gamma(\text{th})}$ with $\gamma(\text{th})$ equal to, respectively, 0.42 (th = 1.5), 0.37 (th = 2.0), 0.34 (th = 2.5).

and included in the interval [0.34,0.42], in agreement with the analogous results found in the financial data [32,33]. This shows also the presence of memory effects induced by noise, in addition to the correlations already pointed out by the deviations from Gaussian behavior quantified by the entropic index q.

In conclusion, we have studied the effect of a small additive noise on a synchronized linear chain of N locally coupled logistic maps at the edge of chaos. We found strong intermittent fluctuations in the returns, whose pdfs are well fitted with q Gaussians. The corresponding interoccurrence times for a fixed threshold exhibit strong analogies with the financial data. This behavior could bring interesting insights on the several common features of real systems of different natures which often operate at the edge of chaos and in weakly noisy environments. The study of the further details of this phenomenon in various complex systems, including earthquakes, is in progress and will be reported elsewhere.

A.P. and A.R. would like to thank CBPF for the warm hospitality and the financial support received in Rio de Janeiro during the preparation of this work. We also thank Alessio Luca, Cigdem Yalcin, Salvo Rizzo, and Bruno Zerbo for useful discussions. Partial financial support from CNPq and Faperj (Brazilian agencies) is also acknowledged.

[1] S. H. Strogatz, *Sync: The Emerging Science of Spontaneous Order* (Hyperion, New York, 2004).
[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, England, 2001).
[3] J. A. Acebron, L. L. Bonilla, C. J. Perez Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005).
[4] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, New York, 1984).
[5] G. Miritello, A. Pluchino, and A. Rapisarda, Physica A **388**, 4818 (2009).
[6] K. Kaneko, *Simulating Physics with Coupled Map Lattices* (World Scientific, Singapore, 1990).
[7] C. Langton, Physica D **42**, 12 (1990).
[8] S. Kauffman, *The Origins of Order: Self-Organization and Selection in Evolution* (Oxford University Press, New York, 1993).
[9] D. Stokic, R. Hanel, and S. Thurner, Phys. Rev. E **77**, 061917 (2008).
[10] R. Hanel, M. Pöchacker, M. Schölling, and S. Thurner, Plos One **7**, e36679 (2012).
[11] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[12] R. N. Mantegna and B. Spagnolo, Phys. Rev. Lett. **76**, 563 (1996).
[13] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).
[14] F. Caruso, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **105**, 190501 (2010).
[15] J. M. R. Parrondo and L. Dinis, Contemp. Phys. **45**, 147 (2004).
[16] J. B. Satinover and D. Sornette, Eur. Phys. J. B **60**, 369 (2007).
[17] A. Pluchino, A. Rapisarda, and C. Garofalo, Physica A **389**, 467 (2010).
[18] A. Pluchino, A. Rapisarda, and C. Garofalo, Physica A **390**, 3496 (2011).
[19] A. Pluchino, C. Garofalo, A. Rapisarda, S. Spagano, and M. Caserta, Physica A **390**, 3944 (2011).
[20] T. Shibata, T. Chawanya, and K. Kaneko, Phys. Rev. Lett. **82**, 4424 (1999).
[21] J. Teramae and Y. Kuramoto, Phys. Rev. E **63**, 036210 (2001).
[22] N. B. Ouchi and K. Kaneko, Chaos **10**, 359 (2000).
[23] C. Li and J. Fang, *International Symposium on Circuits and Systems (ISCAS), 23–26 May 2005, Kobe, Japan* (IEEE, 2005), pp. 288–291.
[24] A. Robledo, Europhys. News **36**, 214 (2005).
[25] U. Tirnakli, C. Tsallis, and C. Beck, Phys. Rev. E **79**, 056209 (2009).
[26] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra, and A. Rapisarda, Phys. Rev. E **75**, 055101 (2007).
[27] B. Bakar and U. Tirnakli, Phys. Rev. E **79**, 040103 (2009).
[28] A. Celikoglu and U. Tirnakli, Phys. Rev. E **83**, 021124 (2010).
[29] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World*, (Springer, New York, 2009).
[30] A. Pluchino, A. Rapisarda, and C. Tsallis, Physica A **387**, 3121 (2008).
[31] C. Tsallis, A. Rapisarda, A. Pluchino, and E. P. Borges, Physica A **381**, 143 (2007).
[32] M. I. Bogachev and A. Bunde, Phys. Rev. E **78**, 036114 (2008).
[33] J. Ludescher, C. Tsallis, and A. Bunde, Europhys. Lett. **95**, 68002 (2011).