Noise, Synchrony and Correlations at the Edge of Chaos

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We study the effect of a weak random additive noise in a linear chain of N locally-coupled logistic maps at the edge of chaos. Maps tend to synchronize for a strong enough coupling, but if a weak noise is added, very intermittent fluctuations in the returns time series are observed. This intermittency tends to disappear when noise is increased. Considering the pdfs of the returns, we observe the emergence of fat tails which can be satisfactorily reproduced by q-Gaussians curves typical of nonextensive statistical mechanics. Interoccurrence times of these extreme events are also studied in detail. Similarities with recent analysis of financial data are also discussed.

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Since the nonlinear phenomenon of synchronization was first observed and discussed in the 17th century by Huygens with clocks, it has become of fundamental importance in various fields of science and engineering. It is frequently observed in complex systems such as biological ones, or single cells, physiological systems, organisms and even populations. Synchrony among coupled units has been extensively studied in the past decades providing important insights on the mechanisms that generate such an emergent collective behavior [1–6]. In this context coupled maps have often been used as a theoretical model (e.g., see [7]). Actually many biological complex systems operate frequently at the edge of chaos and in a noisy environment. Therefore studying the effect of a weak noise in this kind of coupled systems could be relevant in order to understand the way in which interacting units behave in reality. Generally speaking, random noise is considered a disturbance, i.e. something to avoid in order to obtain precise measurements or to minimize numerical errors. But, if on one hand it is very difficult to completely eliminate the effect of noise, on the other hand it can frequently have even a beneficial role. Among the many examples in physics and biology we may cite stochastic resonance [8, 9], noise enhanced stability [10], induced second-order like phase transitions [11] or enhanced diffusion in communication networks [12]. Recently random strategies have been demonstrated to be very successful also in minority and Parrondo games [13–16] and in sociophysics models related to efficiency in hierarchical organizations [17, 18] or even in Parliament models [19]. In this letter we investigate the role of noise in a model of coupled logistic maps at the edge of chaos. More precisely, we study a linear chain of N locally coupled logistic maps following ref. [7] and we explore the role that a small random noise can have in creating a strong intermittent behavior in otherwise synchronized maps. We will also show that the latter behavior can be nicely framed in the context of non extensive statistical mechanics [20, 21], as has also been recently proposed for stock market data analysis [22]. Actually, power-law correlations and intermittent behavior have already been observed in lattices of logistic maps when some kind of global coupling exist among them, see for example refs.[23, 24]. In the latter, in particular, a small-world network of chaotic maps was considered and the occurrence of *on-off* intermittency was due to the long range correlations among maps induced by the 'weak ties' present in the network.

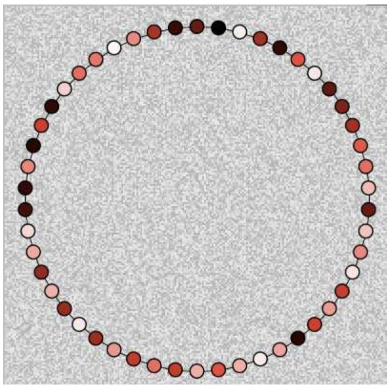

FIG. 1: A pictorial view of a chain of $N = 50$ locally coupled logistic maps embedded in a noisy environment. The different colors indicate, at a fixed time t, different values of the maps in the interval $[-1, 1]$.

In previous studies [25, 26] the effects of a small noise on globally coupled chaotic units was presented for several kind of systems and a universal behavior related to the Lyapunov spectrum was found to be a common feature. In our study, on the contrary, we consider only local coupling and long-range correlations are induced by the noisy environment in which our maps are embedded. Furthermore, in our case maps are not in a chaotic regime, but at the edge of chaos, where the Lyapunov exponent

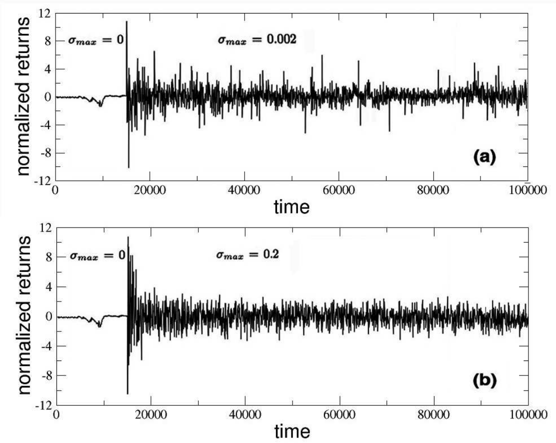

FIG. 2: We show the effect of noise in the normalized returns of Eq. (2) for the case $N = 100$, $\mu = \mu_c = 1.4011551...$, $\epsilon = 0.8$ and $\tau = 32$ time steps. During the first 15.000 time steps at zero noise ($\sigma_{max} = 0$) the maps remain synchronized due to the strong coupling. At time $t = 15000$ we switch on the noise, with $\sigma_{max} = 0.002$ in panel (a) and $\sigma_{max} = 0.2$ in panel (b). A clear intermittent behavior emerges only for the case of weak noise (a), while it is absent for the case of strong noise (b). See text for further details.

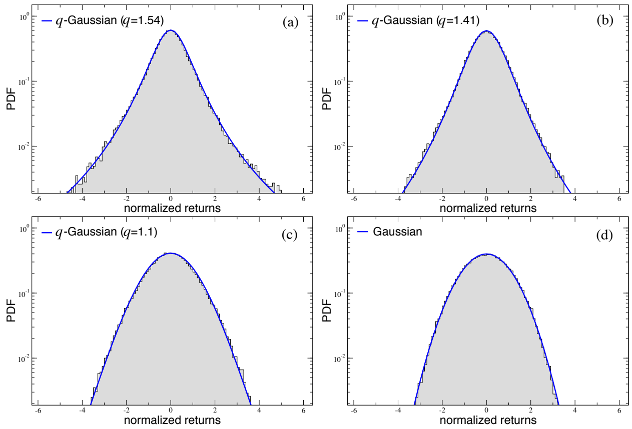

FIG. 3: Pdfs of the returns for $N = 100$ maps at the edge of chaos, i.e. $\mu_c = 1.4011551...$, with $\epsilon = 0.8$, $\tau = 32$ and for increasing values of the noise: $\sigma_{max} = 0.002$ (a), $\sigma_{max} = 0.01$ (b), $\sigma_{max} = 0.05$ (c) and $\sigma_{max} = 0.3$ (d). Fat tails are more evident for weak noise and tend to diminish by increasing noise. We report also fits of the simulation data (full curve) by means of q-Gaussian curves with values $q = 1.54$, $q = 1.41$, $q = 1.1$ and $q = 1$ (corresponding to a Gaussian) respectively. Returns are also normalized to the standard deviation in order to have pdf with unit variance. See text for further details.

is vanishing. In this respect our model presents new features never addressed before as far as we know and more related to realistic complex systems. In the following we describe the model and its dynamics and then discuss the main numerical and analytical results. Connections to non extensive statistics and financial data analysis are then addressed before drawing some conclusions.

We consider a model of a linear chain of N logistic maps locally coupled as

$$x^i_{t+1} = (1-\epsilon)\, f\left({x_t}^i\right) + \frac{\epsilon}{2}\left[f\left({x_t}^{i-1}\right) + f\left({x_t}^{i+1}\right)\right] + \sigma(t) \quad (1)$$

where $\epsilon \in [0, 1]$ is the strength of the local coupling of each map with its first neighbors sites on the chain and the additive noise $\sigma(t)$ is a random variable, fluctuating in time but equal for all the maps, uniformly extracted in the range $[0, \sigma_{max}]$. In our case the i-th logistic map at time t is in the form $f\left(x^i_t\right) = 1 - \mu\left(x^i_t\right)^2$, with $\mu \in [0, 2]$ and with $f\left(x^i_t\right)$ taken in module 1 with sign. The system has periodic boundary conditions. See Fig.1 for a pictorial view.

In the absence of noise, this model was extensively studied by Kaneko et al. [7], in particular in the chaotic regime, where the coupled maps show different patterns of synchronization as function of the coupling strength ϵ. Here we consider the effect of a variable addition of noise on the same system, but at the edge of chaos, i.e. at $\mu_c = 1.4011551...$. Following a procedure adopted in ref.[24], in order to subtract the synchronized component and keep the desynchronized part of each map we consider, at every time step, the difference between the average $< x^i_t >$ and the single map value x^i_t.

Then we further consider the average of the absolute values of these differences over the whole system, i.e.

$$d_t = \frac{1}{N}\Sigma^N_{i=1}|x^i_t - < x^i_t > | \quad (2)$$

in order to measure the distance from the synchronization regime at time t. If all maps are trapped in some synchronized pattern then this quantity remains close to zero, otherwise oscillations are found. As commonly used in turbulence or in finance [21, 22], we analyze these oscillations by considering the two-time returns Δd_t with an interval of τ time steps, defined as follows

$$\Delta d_t = d_{t+\tau} - d_t \quad (3)$$

In Fig.2 we show that this quantity is very sensitive to the noise intensity. In panels (a) and (b) we plot the time evolution of Δd_t (normalized to the standard deviation of the overall sequence) for two different simulations obtained with a linear chain of $N = 100$ maps, with $\epsilon = 0.8$, $\tau = 32$ and considering the maps at the edge of chaos. For both the simulations we consider a transient of 15.000 iterations, during which the system evolves in the absence of noise ($\sigma_{max} = 0$), then we suddenly increase the level of noise bringing it on at $\sigma_{max} = 0.002$ (a) and $\sigma_{max} = 0.2$ (b) respectively: it clearly appears that only in presence of weak noise (panel (a)) the returns time series shows large deviations from the synchronized pattern of the transient, while a higher noise intensity destroys the intermittency and induces Gaussian fluctuations.

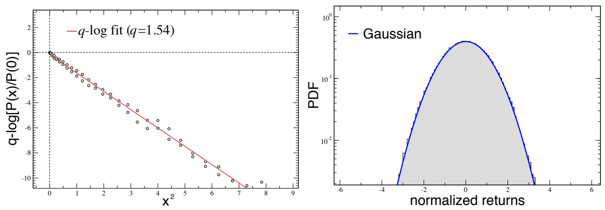

FIG. 4: Left panel: The q-logarithm of the pdf reported in Fig.3(a) (normalized to the peak) is plotted as function of x^2. A q-logarithmic curve with $q = 1.54$ fits the points with a correlation coefficient equal to 0.9958, which indicates that the q-logarithm reproduces accurately the simulation points. See text for further details. Right panel: Same simulation reported in Fig.3(a), but this time with the maps in the fully chaotic regime ($\mu = 2.0$) instead of being at the edge of chaos. A Gaussian shape of the pdf indicates that in this case noise is no longer able to induce intermittency and correlations in the system.

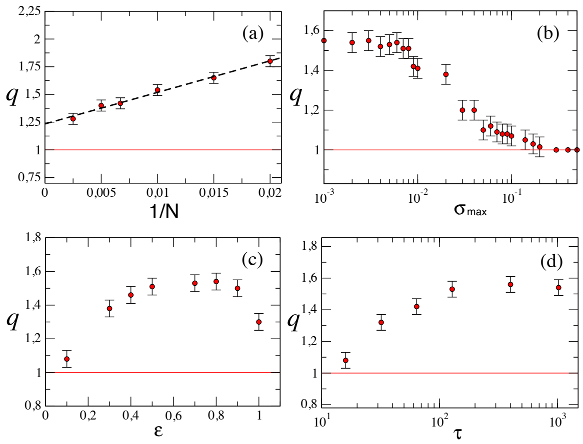

FIG. 5: A summary of the main results found in our study as a function, respectively, of: the size of the system N (a), the level of noise σ_{max} (b), the coupling ϵ (c) and the returns interval τ (d). The maps were always considered at the edge of chaos. See text for further details.

In order to quantify such a different noise-dependent behavior we plot in Fig.3 the probability distribution function (pdf) of normalized returns for increasing values of σ_{max}, from 0.002 to 0.3. Fat tails in the pdfs are visible only for $\sigma_{max} < 0.1$ and can be very well reproduced with q-Gaussian curves, typical of non extensive statistical mechanics [21]. These curves are defined as

$$G_q(x) = A\left[1 - (1-q)\,\beta x^2\right]^{\frac{1}{1-q}}, \qquad (4)$$

where q is the entropic index (which evaluates deviations from Gaussian behavior), $1/\beta$ plays the role of a variance and A is a normalization parameter. For $\sigma_{max} = 0.002$ (panel (a)), where the tails are very pronounced, one has $q \sim 1.5$ while, for higher values of noise, the tails tend to disappear and the value of q decreases asymptotically towards $q = 1$, which corresponds to a Gaussian pdf (panel (d), with $\sigma_{max} = 0.3$). This definitively demonstrates that if some noise creates intermittency and correlations, too much noise destroys them. As further test to verify the accuracy of the q-Gaussian fit (Eq. (4)) shown in Fig.3(a), in the left panel of Fig.4 we plot (as open circles) the q-logarithm (defined as $\ln_q z \equiv [z^{1-q}-1]/[1-q]$, with $\ln_1 z = \ln z$) of the corresponding pdf, normalized to its peak, as function of x^2, and we verify that a q-logarithm curve with $q = 1.54$ fits very well the simulation points with a correlation coefficient equal to 0.9958. Finally, in the right panel of Fig.4, we show that the Gaussian behavior of Fig.3(d) can be also obtained considering the same parameters of Fig.3(a), but with the maps in the fully chaotic regime, i.e. with $\mu = 2.0$ instead of $\mu = \mu_c$. This indicates that the edge of chaos condition is strictly necessary for the emergence of intermittency and strong correlations in presence of a small level of noise.

If one considers the value of the entropic index q, emerging through q-Gaussian fits of the returns Pdfs, as a measure of the correlations induced by the noisy environment on our chain of coupled maps at the edge of chaos, it is worthwhile to explore how this value changes as function, not only of the noise σ_{max}, but also of the number N of maps, the coupling strength ϵ and the returns time interval τ. We show in Fig.5 a summary of the results obtained in this direction for $\mu = \mu_c = 1.4011551...$ and changing the parameters $\sigma_{max} = 0.002$, $N = 100$, $\epsilon = 0.8$ and $\tau = 32$ one at a time and then calculating the corresponding values of q as reported. More precisely, in panel (a) we plot the entropic index as function of $1/N$ and we see that q remains greater than 1 also for very large N, thus implying that the noise induced correlations are not a finite-size effect. The influence of noise on the value of q used to fit the pdf of the returns is reported in panel (b), where an asymptotic convergence towards 1, for strong noise, and towards ~ 1.5, for weak noise, is clearly visible. Quite interestingly, the plot of q as function of the coupling strength, panel (c), has a maximum in correspondence of $\epsilon \sim 0.8$, a value which evidently allows an optimal spreading of correlations over the maps chain in the presence of a small noise. Finally, in panel (d) we plot q versus the time interval τ used to calculate the returns: the resulting points seem to saturate for high values of τ, thus confirming the robustness of this kind of correlations for low levels of noise.

Long-term correlations in a system typically yield power-law asymptotic behaviors in various physically relevant properties. In studies of financial markets, it was recently observed [22] power-law decays in the so-called 'interoccurrence times' between sub sequential peaks in the fluctuating time series of returns like those shown in Fig.2(a). If we fix a given threshold, the sequence of the interoccurrence time intervals results to be well defined and it is then possible to study its pdf. We do this for our

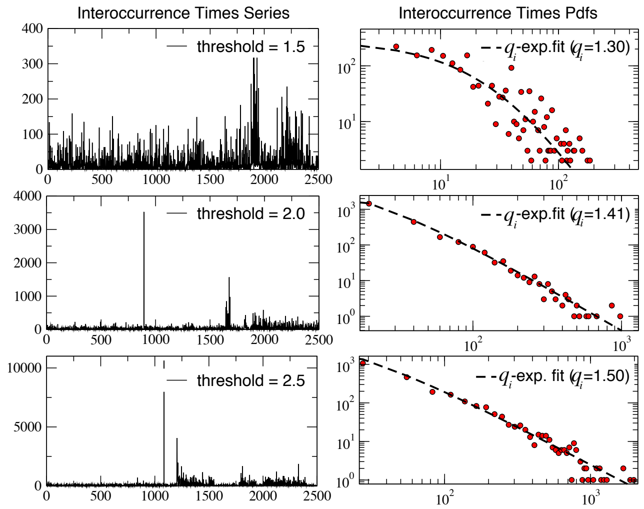

FIG. 6: Left column panels: plots of the interoccurrence times τ_i of returns for increasing thresholds in the case $N = 100$, $\epsilon = 0.8$, $\sigma_{max} = 0.002$ and $\tau = 32$. Right column panels: Pdf of the time series reported on the left panels. These pdfs are nicely fitted by q_i-exponential curves, whose value of q_i is also reported. See text for further details.

usual chain of $N = 100$ maps at the edge of chaos, with $\epsilon = 8$, $\tau = 32$ and for a weak noise with $\sigma_{max} = 0.002$. In the left panels of Fig.6, the interoccurrence time series for the normalized returns are plotted (from top to bottom) in correspondence of three increasing values of the threshold (1.5, 2.0 and 2.5), while the correspondent pdfs are reported on the right. In all the cases q_i-exponentials (i.e., Pdf $\propto [1-(1-q_i)\tau_i/\tau_{q_i}]^{1/1-q_i}$, where the subindex i stands for *interoccurrence*) satisfactorily fit the data for values of q_i which depend on the threshold, in complete analogy with what was observed for financial data [22]. In Fig.7 we also show that q_i scales linearly as function of the threshold. This can be considered as a further footprint of the complex emergent behavior induced on the system by the small level of noise considered. Interestingly enough, in the limit of vanishing threshold, q_i approaches unity, i.e., the behavior becomes exponential, which is precisely what was systematically observed in financial data [22].

In conclusion, we have studied the effect of a small additive noise on a synchronized linear chain of N locally-coupled logistic maps at the edge of chaos. We found strong intermittent fluctuations in the returns, whose pdfs are well fitted with q-Gaussians. The corresponding interoccurrence times for fixed threshold exhibit strong analogies with financial data[22]. This behavior could bring interesting insights on the several common features of real systems of different nature which often operate at the edge of chaos and in weakly noisy environments. The maximum in Fig.5(c) corresponds to the fattest tails in the Pdf's, i.e., frequent large jumps. The fact that they

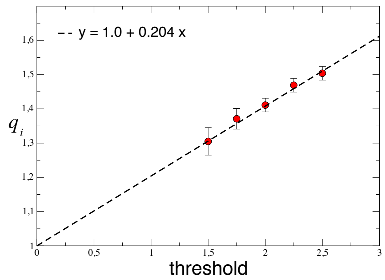

FIG. 7: The values of the index q_i, resulting by q_i-exponential fits of the interoccurrence time series Pdfs, are reported in correspondence of five values of the threshold (1.5, 1.75, 2.0, 2.25, 2.5). A linear fit is also plotted for comparison.

occur for finite levels of ϵ is somewhat reminiscent of phenomena such as stochastic resonance [8, 9]. The study of further details of this behavior in various complex systems, including earthquakes, is in progress and will be reported elsewhere.

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