Stationary and uniformly accelerated states in nonlinear quantum mechanics

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We consider two kinds of solutions of a recently proposed field theory leading to a nonlinear Schrödinger equation exhibiting solitonlike solutions of the power-law form $e_q^{i(kx-wt)}$, involving the q exponential function naturally arising within nonextensive thermostatistics [$e_q^z \equiv [1+(1-q)z]^{1/(1-q)}$, with $e_1^z = e^z$]. These fundamental solutions behave like free particles, satisfying $p = \hbar k$, $E = \hbar\omega$, and $E = p^2/2m$ ($1 \leqslant q < 2$). Here we introduce two additional types of exact, analytical solutions of the aforementioned field theory. As a first step we extend the theory to situations involving a potential energy term, thus going beyond the previous treatment concerning solely the free-particle dynamics. Then we consider both bound, stationary states associated with a confining potential and also time-evolving states corresponding to a linear potential function. These types of solutions might be relevant for physical applications of the present nonlinear generalized Schrödinger equation. In particular, the stationary solution obtained shows an increase in the probability for finding the particle localized around a certain position of the well as one increases q in the interval $1 \leqslant q < 2$, which should be appropriate for physical systems where one finds a low-energy particle localized inside a confining potential.

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I. INTRODUCTION

Nonlinear partial differential equations constitute important tools for the description of a wide family of physical systems and processes [1–3]. Among the most intensively investigated nonlinear differential equations we have the nonlinear versions of the Schrödinger [3–6] and the Fokker-Planck [7–11] equations. Here we focus primarily on a recently advanced nonlinear Schrödinger equation (NLSE) [4] that may be related to nonextensive statistical mechanics and the associated nonadditive entropies [12–14].

The free-particle NLSE proposed in [4] has a formal resemblance to the power-law nonlinear diffusion equation and to the associated nonlinear Fokker-Planck equation [7], which has been successfully applied to the study of a variety of subjects ranging from diffusion in porous media to processes in finance [15]. This resemblance is, of course, already apparent in the $q \to 1$ linear limit. In fact, the close structural connection between the linear Schrödinger and the Fokker-Planck equations [16] suggests an exploration of the dynamics of a complex Schrödinger-like counterpart of the power-law nonlinear diffusion equation. This complex equation is obtained by replacing the diffusion (real) constant by a (purely imaginary) complex number. As happens in the linear scenario, this modification gives rise to profound changes in the associated dynamics. The main new feature exhibited by the nonlinear evolution equation introduced in [4] is that it admits solitonlike solutions where the space-time dependence of the wave function $\Psi(x,t)$ occurs solely through a single variable of the form $x - vt$, corresponding to a space displacement at a constant velocity v without change in the wave function's shape. These solitonlike solutions are known as q plane waves and are compatible with the Planck and de Broglie relations, satisfying $E = \hbar w$ and $p = \hbar k$, with $E = p^2/2m$. Furthermore, under Galilean transformations the q plane waves recover the transformations rules of the linear Schrödinger equation [6]. The NLSE satisfied by the q plane waves can be obtained from a field theory based upon an action variational principle [5]. These properties suggest that the q-plane-wave solutions of the NLSE can be regarded as a new field-theoretical description of particle dynamics that may be relevant in diverse areas of physics, including nonlinear optics, superconductivity, plasma physics, and dark matter [5,17].

The theoretical framework within which q plane waves emerged generalizes the Boltzmann-Gibbs (BG) entropy and statistical mechanics, through the introduction of an index q ($q \to 1$ recovers the BG case). Considerable progress has been achieved along these lines of research, leading, for instance, to nonlinear extensions of various important equations of physics and new forms of the central limit theorem [18]. Central to these developments are the q Gaussian distributions, which generalize the standard Gaussian distribution and appear naturally when optimizing the q entropy [12], or from the solution of the corresponding nonlinear Fokker-Planck equation [10]. The q Gaussians have been successfully applied to the analysis of recent experimental results in various fields [13]. Among others, we may mention the following: (i) the velocities of cold atoms in dissipative optical lattices [19]; (ii) the velocities of particles in quasi-two-dimensional dusty plasma [20]; (iii) single ions in radio-frequency traps interacting with a classical buffer gas [21]; (iv) the relaxation curves of Ruderman-Kittel-Kasuya-Yosida (RKKY) spin glasses, like CuMn and AuFe [22]; (v) transverse momenta distributions in Large Hadron Collider (LHC) experiments [23]. Recent progress in the study of the dynamics given by the nonlinear Schrödinger

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equation advanced in [4] includes the investigation of its behavior under Galilean transformations [6], of q Gaussian wave packet solutions [24], of quasistationary solutions [25], and of its connection with the Bohmian approach to quantum mechanics [26].

It was shown in [5] (see also [27]) that, besides the wave function $\Psi(x,t)$ and its associated evolution equation, an extra field $\Phi(x,t)$ must be introduced satisfying a differential equation coupled with the one originally advanced in [4]. The coupled nonlinear equations jointly governing the dynamics of the fields Ψ and Φ are then

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right] &= -\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{2-q}, \\ i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] &= \frac{\hbar^2}{2m}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{1-q}\nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right], \end{aligned} \tag{1}$$

where $q \geqslant 1$ and the real, positive constants Ψ_0 and Φ_0 guarantee the correct physical dimensionalities for all terms (notice that this scaling becomes irrelevant only for the linear equation, i.e., $q = 1$). The constants Ψ_0 and Φ_0 have to be regarded as parameters characterizing the evolution equation (1) itself (that is, not as part of the initial conditions). The first of the equations (1) may be seen as the master equation, whereas the second is the slave.

Consistently with the evolution equations (1), for any q and an arbitrary finite volume Ω, the probability density for finding a particle at time t in a given position x can be defined as

$$\rho(\vec{x},t) = \frac{1}{2\Omega\Psi_0\Phi_0}[\Psi(\vec{x},t)\Phi(\vec{x},t) + \Psi^*(\vec{x},t)\Phi^*(\vec{x},t)]. \tag{2}$$

In general, this density satisfies the balance equation,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = R, \tag{3}$$

where

$$\begin{aligned} \vec{J} = \frac{i\hbar}{4m\Omega\Psi_0^{2-q}\Phi_0}&[-\Psi^{1-q}(\vec{\nabla}\Psi)\Phi + (\Psi^*)^{1-q}(\vec{\nabla}\Psi^*)\Phi^* \\ &+ \Psi^{2-q}(\vec{\nabla}\Phi) - (\Psi^*)^{2-q}(\vec{\nabla}\Phi^*)], \end{aligned} \tag{4}$$

and

$$\begin{aligned} R = \frac{i(1-q)\hbar}{4m\Omega\Psi_0^{2-q}\Phi_0}&[\Psi^{1-q}(\vec{\nabla}\Psi)\cdot(\vec{\nabla}\Phi) \\ &- (\Psi^*)^{1-q}(\vec{\nabla}\Psi^*)\cdot(\vec{\nabla}\Phi^*)]. \end{aligned} \tag{5}$$

Note that in the limit $q \to 1$ the second equation in (1) reduces to the complex conjugate of the first one. That is, when $q \to 1$ the function $\Phi(x,t) = \Psi^*(x,t)$, with $\Psi(x,t)$ a solution of the first equation in (1), satisfies the second field equation in (1). It is interesting to note that for the q-plane-wave solutions of the equation governing the field $\Psi(x,t)$, the field $[\Phi(x,t)/\Phi_0] = [\Psi(x,t)/\Psi_0]^{-q}$ is a solution of the second equation in (1), yielding the probability density $\rho(x,t) = \frac{1}{\Omega}$ for all values of q, and consequently satisfying a continuity equation.

In spite of the apparent singularity exhibited by the first equation in (1) when $q = 2$, it is easy to verify that this equation is well defined for this value of q. It should be emphasized that the coupled equations (1) have important structural differences with respect to other nonlinear Schrödinger equations appearing in the literature, which contain extra nonlinear terms (usually a cubic nonlinearity) involving only the wave function itself. In particular, and most importantly, the first evolution equation in (1) here is endowed with a power-law nonlinearity within the Laplacian term. This equation can be regarded as a q-generalized Schrödinger equation that describes a free particle of mass m moving in d spatial dimensions [4].

As already mentioned, the field equations (1) admit q-plane-wave solutions that are compatible with the Planck and de Broglie relations and behave like free particles. These solutions are expressed in terms of the q-exponential function $\exp_q(u)$ which, for a purely imaginary iu, is defined as the principal value of

$$\begin{aligned} \exp_q(iu) &= [1 + (1-q)iu]^{1/(1-q)}, \\ \exp_1(iu) &\equiv \exp(iu). \end{aligned} \tag{6}$$

The above function satisfies [28]

$$\begin{aligned} \exp_q(\pm iu) &= \cos_q(u) \pm i\,\sin_q(u), \\ \cos_q(u) &= r_q(u)\cos\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}, \\ \sin_q(u) &= r_q(u)\sin\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}, \\ r_q(u) &= [1 + (1-q)^2u^2]^{1/[2(1-q)]}, \end{aligned} \tag{7}$$

$$\begin{aligned} &\exp_q(iu)\exp_q(-iu) = [r_q(u)]^2 = \exp_q[-(q-1)u^2], \\ &\exp_q(iu_1)\exp_q(iu_2) \neq \exp_q[i(u_1+u_2)] \quad (q \neq 1). \end{aligned} \tag{8}$$

As a consequence of Eqs. (7) and (8), a q exponential with a purely imaginary argument, $\exp_q(iu)$, presents an oscillatory behavior with a u-dependent amplitude $r_q(u)$. The function $\exp_q(iu)$ complies with the physically important property of square integrability for $1 < q < 3$, whereas the concomitant integral diverges in both limits $q \to 1$ and $q \to 3$ and also for $q < 1$ [29].

The d-dimensional q-plane-wave solution of Eqs. (1) is given by

$$\begin{aligned} \Psi(\vec{x},t) &= \Psi_0\,\exp_q[i(\vec{k}\cdot\vec{x} - \omega t)], \\ \Phi(\vec{x},t) &= \Phi_0\,\exp_q^{-q}[i(\vec{k}\cdot\vec{x} - \omega t)]. \end{aligned} \tag{9}$$

If we take into account that $d\exp_q(z)/dz = [\exp_q(z)]^q$ and $d^2\exp_q(z)/dz^2 = q[\exp_q(z)]^{2q-1}$ we obtain, for the d-dimensional Laplacian,

$$\begin{aligned} \nabla^2\left(\frac{\Psi}{\Psi_0}\right)^\alpha &= -\alpha(\alpha+q-1)\left(\sum_{n=1}^{d}k_n^2\right)\left(\frac{\Psi}{\Psi_0}\right)^{\alpha+2q-2}, \\ \nabla^2\left(\frac{\Phi}{\Phi_0}\right) &= -q\left(\sum_{n=1}^{d}k_n^2\right)\left(\frac{\Phi}{\Phi_0}\right)^{q-2}, \end{aligned} \tag{10}$$

with α any real constant. Now, inserting the q-plane-wave ansatz (9) into the nonlinear field equations (1), we verify that the q plane wave is indeed a solution provided that the frequency ω and the momentum k satisfy the relation $\omega = \frac{\hbar k^2}{2m}$.

Equivalently, if one makes [4], according to the celebrated de Broglie and Planck relations, the identifications $\vec{k} \to \vec{p}/\hbar$ and $\omega \to E/\hbar$, one verifies that the q plane wave is a solution of Eq. (1) with $E = p^2/2m$, thus *preserving the energy spectrum of the free particle for all values of* q. Therefore (1), together with its solution Eq. (9), can be considered as candidates for describing interesting types of physical phenomena.

The q-plane-wave solutions (9) have been so far the only known solutions of the coupled nonlinear field equations (1). In the case of the q-generalized NLSE [the first equation in (1)] other exact solutions besides q plane waves have been recently discovered. For instance, a family of solutions to the NLSE has been obtained exhibiting the form of q Gaussian wave packets [24]. Needless to say, an important feature of the linear Schrödinger equation is the superposition principle, so that one can express the solution as a linear combination of other functions. However, for the NLSE, the superposition principle is not satisfied by all solutions. In this context, it is very relevant to explore a considerable number of different solutions for the nonlinear equations.

The purpose of the present work is twofold. We are going to consider the extension of the nonlinear field theory proposed in [5] beyond the free-particle dynamics, and to study additional families of exact solutions of the associated field equations. In particular, we shall obtain solutions which can be decomposed into spatial and temporal parts. It is easy to see that the q plane wave cannot be decomposed in such a way when $q \neq 1$. Using the method of separation of variables we find different ordinary differential equations for the two fields appearing in (1). Furthermore, we are going to discuss a different family of exact analytical solutions of Eqs. (1) corresponding to a linear potential function.

II. PARTICLE IN A POTENTIAL

The generalization of the Schrödinger equation proposed in Ref. [4] was formulated only for free particles. The q-plane-wave solution expresses a possible dynamics of a particle of mass m in a space in the absence of external potentials. Symmetry considerations applied to this formalism [6] have shown important features allowing this theory to be extended beyond free-particle dynamics. For example, consistent with what happens in the standard Schrödinger equation, the behavior of a free particle in the NLSE in the presence of uniform acceleration can be interpreted as describing a free particle under a constant force. However, instead of the usual coupling between the potential and the function ψ, within the NLSE formalism, the potential is attached to ψ^q [6].

Following the main conclusions derived from the analysis conducted in [6], it is possible to reformulate the field theory introduced in [5] in order to incorporate appropriate potential energy terms in the associated field equations and thus extend the theory beyond free-particle dynamics. This allows for discussing the dynamics of a particle in the presence of an external potential $V(x)$. The appropriate Lagrangian density is given by

$$L = L_{\text{free}} - V(x)\tilde{\Phi}(x,t)\tilde{\Psi}(x,t)^q - V(x)\tilde{\Phi}^*(x,t)\tilde{\Psi}^*(x,t)^q, \tag{11}$$

where L_{free} is the Lagrangian density of the free particle [5], $\tilde{\Psi} = \Psi/\Psi_0$ and $\tilde{\Phi} = \Phi/\Phi_0$.

From the above equation, using the Euler-Lagrange equation for each field, we obtain the NLSE

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Psi(x,t)}{\Psi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\frac{\Psi(x,t)}{\Psi_0}\right]^{2-q} + V(x)\left[\frac{\Psi(x,t)}{\Psi_0}\right]^q, \tag{12}$$

together with the new field equation

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = \frac{\hbar^2}{2m}\left[\frac{\Psi(x,t)}{\Psi_0}\right]^{1-q}\frac{\partial^2}{\partial x^2}\left[\frac{\Phi(x,t)}{\Phi_0}\right] - qV(x)\left(\frac{\Psi(x,t)}{\Psi_0}\right)^{q-1}\left[\frac{\Phi(x,t)}{\Phi_0}\right]. \tag{13}$$

An interesting feature of the field equations (12) and (13) is that in the first equation the potential V couples to Ψ^q, instead of coupling to Ψ, as happens in the standard linear Schrödinger equation ($q = 1$). In the case of the second field equation (13) the potential couples to $\Psi^{(q-1)}\Phi$. In this regard it is interesting to note that the q-plane-wave solution given by the fields $\Psi(x,t) = \Psi_0 \exp_q[i(kx - \omega t)]$ and $\Phi(x,t) = \Phi_0 \exp_q[i(kx - \omega t)]^{-q}$ is a solution not only of the nonlinear field equations corresponding to a free particle (with $\hbar\omega = \frac{\hbar^2 k^2}{2m}$) but also of the nonlinear equations

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left[\frac{\Psi}{\Psi_0}\right] &= -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\frac{\Psi}{\Psi_0}\right]^{2-q} + V_0\left[\frac{\Psi}{\Psi_0}\right]^q, \\ i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi}{\Phi_0}\right] &= \frac{\hbar^2}{2m}\left[\frac{\Psi}{\Psi_0}\right]^{1-q}\frac{\partial^2}{\partial x^2}\left[\frac{\Phi}{\Phi_0}\right] - q\left[\frac{\Phi}{\Phi_0}\right]V_0\left[\frac{\Psi}{\Psi_0}\right]^{q-1}, \end{aligned} \tag{14}$$

with a constant potential V_0, provided that $\hbar\omega = \frac{\hbar^2 k^2}{2m} + V_0$, which, using the Planck and de Broglie relations, becomes $E = \frac{p^2}{2m} + V_0$, as expected.

The field equations (12) and (13) still have an associated density ρ of the form (2), which satisfies the balance equation (3) with the probability density current (4) and

$$\begin{aligned} R = &\frac{i(1-q)\hbar}{4m\Omega\Psi_0^{2-q}\Phi_0}[\Psi^{1-q}(\vec{\nabla}\Psi)\cdot(\vec{\nabla}\Phi) \\ &- (\Psi^*)^{1-q}(\vec{\nabla}\Psi^*)\cdot(\vec{\nabla}\Phi^*)] \\ &+ \frac{(1-q)}{2i\hbar\Omega\Psi_0^q\Phi_0}[\Psi^q\Phi - (\Psi^*)^q\Phi^*]V(\vec{x}). \end{aligned} \tag{15}$$

Note that the presence of the potential $V(\vec{x})$ gives rise to a new term in R not appearing in the previous expression (5). As a consequence of the NLSE for a particle in the presence of external potentials, we can observe that decomposed solutions are not admitted anymore. The exception is the infinite potential well that we address in the following section.

III. STATIONARY STATES

Now, we introduce solutions for Eq. (12) that can be decomposed into spatial and temporal parts. Let us start with the case of a free particle, $V(x)=0$; considering $\Psi(x,t)=\psi_1(x)\psi_2(t)$ we obtain

$$-\frac{\hbar^2}{2m(2-q)}\frac{\psi_{10}}{\psi_1(x)}\frac{d^2}{dx^2}\left[\frac{\psi_1(x)}{\psi_{10}}\right]^{2-q}$$
$$=i\hbar\left[\frac{\psi_{20}}{\psi_2(t)}\right]^{2-q}\frac{d}{dt}\left[\frac{\psi_2(t)}{\psi_{20}}\right]=\epsilon, \tag{16}$$

where ϵ, ψ_{10}, and ψ_{20} are constants. Thus, we can write

$$\frac{d^2}{dx^2}\left[\frac{\psi_1(x)}{\psi_{10}}\right]^{2-q}+\epsilon\frac{2m(2-q)}{\hbar^2}\left[\frac{\psi_1(x)}{\psi_{10}}\right]=0 \tag{17}$$

and

$$\frac{d}{dt}\left[\frac{\psi_2(t)}{\psi_{20}}\right]+i\frac{\epsilon}{\hbar}\left[\frac{\psi_2(t)}{\psi_{20}}\right]^{2-q}=0. \tag{18}$$

By direct calculus, we find the expressions

$$\psi_1(x)=\psi_{10}\exp_{(q+1)/2}\left[i\sqrt{\frac{4m\epsilon}{(3-q)\hbar^2}}\,x\right] \tag{19}$$

and

$$\psi_2(t)=\psi_{20}\exp_{2-q}[-i\epsilon t/\hbar]. \tag{20}$$

Now we take into account Eqs. (19) and (20) to obtain

$$\Psi(x,t)=\psi_0\left[\frac{1+(1-q)ikx\sqrt{\frac{2}{3-q}}-(1-q)^2\frac{k^2x^2}{2(3-q)}}{1-(q-1)\frac{i\epsilon t}{\hbar}}\right]^{1/(1-q)}. \tag{21}$$

In particular, we emphasize that in the limit of $(q-1)\to 0$ if we retain only terms of order $q-1$, the present decomposed solution coincides with the family of solutions given by q plane waves $\Psi(x,t)=\psi_0\exp_q[i(kx-wt)]$ in the same limit.

The extra field $\phi(x,t)$ can also be decomposed as $\Phi(x,t)=\phi_1(x)\phi_2(t)$, so using Eq. (13) we obtain

$$-\frac{\hbar^2}{2m}\frac{\phi_{10}}{\phi_1(x)}\left[\frac{\psi_1(x)}{\psi_{10}}\right]^{1-q}\frac{d^2}{dx^2}\left[\frac{\phi_1(x)}{\phi_{10}}\right]$$
$$=-i\hbar\left[\frac{\psi_{20}}{\psi_2(t)}\right]^{1-q}\frac{\phi_{20}}{\phi_2(t)}\frac{d}{dt}\left[\frac{\phi_2(t)}{\phi_{20}}\right]=\mu, \tag{22}$$

which yields $[\phi_1(x)/\phi_{10}]=[\psi_1(x)/\psi_{10}]^{-1}$ and $[\phi_2(t)/\phi_{20}]=[\psi_2(t)/\psi_{20}]^{-1}$, where we have observed that the constant $\mu=\epsilon$ in order that the solution presented here can also be characterized by the probability density $\rho(x,t)=\frac{1}{\Omega}$ for all values of q.

The extension of these results to the d-dimensional case is straightforward. We observe that the solutions of the d-dimensional NLSE can also be decomposed into spatial and temporal parts. In contrast to the linear case, however, the spatial wave cannot be decomposed into d different coordinate components. In any case, we have $\rho(\vec{r},t)=\frac{1}{\Omega^d}$ for all d.

A. The infinite potential well

An important feature of the decomposed solution concerns the stationarity for the NLSE. In this situation we can investigate a time-independent probability density as in standard quantum mechanics for a state with definite energy spectrum. As an example, we are going to solve the problem of a particle in an infinite rectangular potential well, where

$$V(x)=\begin{cases}0, & 0<x<a,\\ \infty & \text{otherwise.}\end{cases} \tag{23}$$

Notice that the superposition principle cannot be applied here; thus we cannot use the solution $\psi_1(x)$ as given by Eq. (19) to solve Eq. (17) in the presence of the above potential $V(x)$. Boundary conditions assert that $\psi_1(x)$ must be taken purely real. A family of solutions can be obtained by writing Eq. (17) as [30]

$$\frac{d\varphi(x)}{dx}=\sqrt{2}\sqrt{1-\frac{2m\epsilon(2-q)^2\varphi^{(3-q)/(2-q)}}{(3-q)\hbar^2}}, \tag{24}$$

where we have defined $\varphi(x)=[\psi_1(x)/\psi_{10}]^{3-q}$. It is straightforward to obtain

$$\sqrt{2}\left(\frac{2m\epsilon(2-q)^2}{(3-q)\hbar^2}\right)^{(2-q)/(3-q)}x+\delta$$
$$=\int_0^{\varphi(x)/A_q}\frac{dz}{\sqrt{1-z^{(3-q)/(2-q)}}}, \tag{25}$$

where A_q and δ are constants of integration and $|z|<1$. We could not find an analytical solution for this integral. However, defining

$$x\equiv \mathrm{Sin}_q^{-1}(y)=\int_0^y\frac{dz}{\sqrt{1-z^{(3-q)/(2-q)}}}, \tag{26}$$

we can write the relation $y=\mathrm{Sin}_q(x)$, whose period is $4\tau_q$, where

$$\tau_q=\int_0^1\frac{dz}{\sqrt{1-z^{(3-q)/(2-q)}}}$$
$$=\sqrt{\pi}\frac{(2-q)\Gamma((2-q)/(3-q))}{(3-q)\Gamma((7-3q)/(6-2q))}. \tag{27}$$

These equations express a generalization of the trigonometric function, recovered in the limit $q=1$, i.e., $\mathrm{Sin}_1(x)\equiv\sin(x)$. One should notice that this generalization differs from the $\sin_q(x)$ function that is currently used in nonextensive statistics [28]; in particular, one important distinction concerns the fact that $|\mathrm{Sin}_q(x)|\leqslant 1$ for $1\leqslant q<2$. In Fig. 1 we exhibit the function $\mathrm{Sin}_q(x)$ versus x for typical values of q in the interval $1\leqslant q<2$. Since τ_q decreases for increasing q (cf. Fig. 2), the function $\mathrm{Sin}_q(x)$ becomes contracted for higher values of q as shown in Fig. 1.

Observe that we can also define $\mathrm{Cos}_q(y)\equiv\sqrt{1-\mathrm{Sin}_q(y)^2}$, or even in a more general way, $\mathrm{Cos}_q(y)\equiv[1-\mathrm{Sin}_q(y)^{\alpha_q}]^{1/\alpha_q}$. Further analysis of the properties of these functions would be useful, but it is out of the scope of the present study.

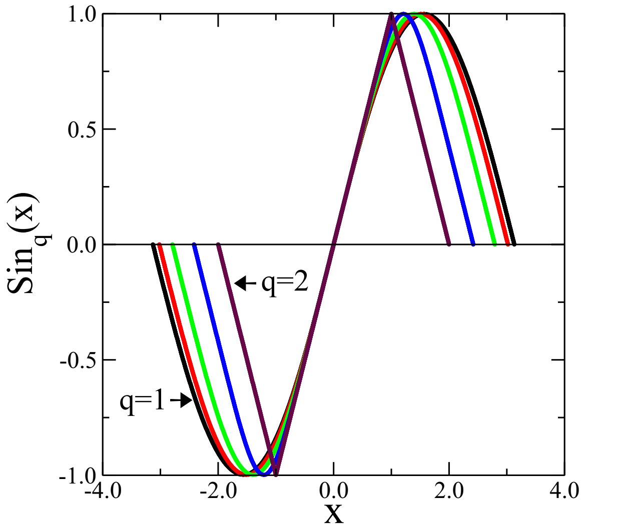

FIG. 1. (Color online) Generalized trigonometric function defined in Eq. (26). The function $\mathrm{Sin}_q(x)$ [$\mathrm{Sin}_1(x) \equiv \sin(x)$] is exhibited versus x for typical values of q, such that in the x interval shown ($x \in [-2\tau_q, 2\tau_q]$ in each case) one has $q = 1$, 1.2, 1.5, 1.8, and 2 (from left to right for $x < -1$), the curves essentially overlap in the interval $-1 < x < 1$, whereas the order gets inverted for $x > 1$. One notices that $|\mathrm{Sin}_q(x)| \leqslant 1$ for $1 \leqslant q < 2$.

Now, considering

$$k_q \equiv \sqrt{2}\left(\frac{2m\epsilon(2-q)^2}{(3-q)\hbar^2}\right)^{(2-q)/(3-q)}, \tag{28}$$

from Eqs. (25) and (26) we have $(k_q x + \delta) = \mathrm{Sin}_q^{-1}[\varphi(x)/A_q]$. Thus $\varphi(x) = A_q \mathrm{Sin}_q(k_q x + \delta)$, and the stationary solution for the NLSE can be written as

$$\psi_1(x) = \psi_{10}[A_q \mathrm{Sin}_q(k_q x + \delta)]^{1/(2-q)}. \tag{29}$$

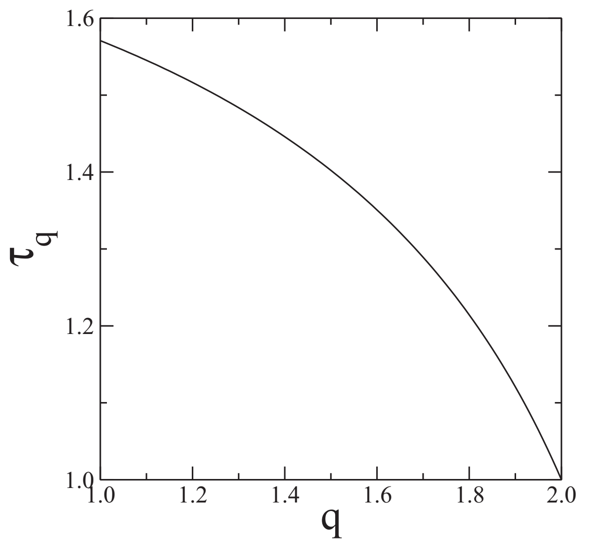

FIG. 2. The dimensionless quantity τ_q [cf. Eq. (27)], associated with the periodicity of the function $\mathrm{Sin}_q(x)$ (which presents a period $4\tau_q$), is exhibited versus q in the interval $1 \leqslant q < 2$, so that one has as particular cases $\tau_1 = \pi/2$ and $\tau_2 = 1$.

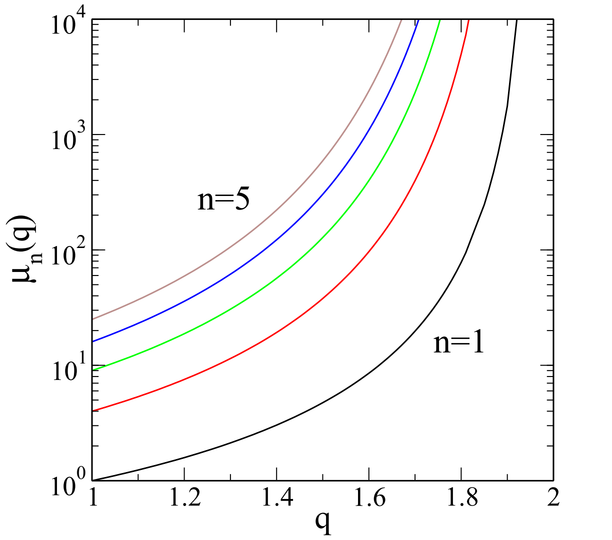

FIG. 3. (Color online) The dimensionless ratio $\mu_n(q) = \epsilon_n(q)/\epsilon_1(1)$ [$\epsilon_1(1) = (\hbar^2/2m)(\pi/a)^2$], for a particle in a one-dimensional infinite potential well of size a [cf. Eq. (23)], is exhibited versus q, for the quantum numbers $n = 1,2,\ldots,5$.

Considering the boundary condition $\psi_1(0) = \psi_1(a) = 0$, we have $\delta = 0$ and $k_q a = 2\tau_q n$, where $n = 1,2,3,\ldots$, so that

$$\psi_1(x) = \psi_1(x,q,n) = \left[\tilde{A}_{q,n}\mathrm{Sin}_q\left(\frac{2n\tau_q x}{a}\right)\right]^{1/(2-q)}. \tag{30}$$

Then the expression

$$\epsilon_n(q) = \frac{(3-q)\hbar^2}{2m(2-q)^2}\left(\frac{\sqrt{2}n\tau_q}{a}\right)^{(3-q)/(2-q)} \tag{31}$$

generalizes the energy spectrum of the standard quantum well, $\epsilon_n(1) = (\hbar^2/2m)(n\pi/a)^2$. In Fig. 3 we present the dimensionless ratio $\mu_n(q) = \epsilon_n(q)/\epsilon_1(1)$ versus q, for the quantum numbers $n = 1,2,\ldots,5$. One notices the same qualitative behavior of $\epsilon_n(q)$ on varying q, for all values of n, and as expected $\epsilon_n(q)$ increases on increasing n for any q in the interval considered. However, one notices that $\epsilon_n(q)$ diverges as $q \to 2$, for all values of n.

The extra field $\phi_1(x)$ is obtained by inserting the solution for $\psi_1(x)$ of Eq. (30) into Eq. (13), so that by choosing $\mu = (2-q)\epsilon$ we find

$$\phi_1(x) = [\psi_1(x)]^{2-q}. \tag{32}$$

Finally, we can write the probability density as

$$\rho(x) = \frac{\mathrm{Re}\{[\mathrm{Sin}_q(2n\tau_q x/a)]^{(3-q)/(2-q)}\}}{a\int_0^1 dx\, \mathrm{Re}\{[\mathrm{Sin}_q(2n\tau_q x/a)]^{(3-q)/(2-q)}\}}, \tag{33}$$

where $\mathrm{Re}\{s\}$ stands for the real part of s and we have used Eq. (2), as well as the normalization condition for finding the amplitude $\tilde{A}_{n,q}$. It is important to stress that $\mathrm{Re}\{[\mathrm{Sin}_q(2n\tau_q x/a)]^{(3-q)/(2-q)}\} > 0$ for $1 < q < 4/3$ and that this quantity may also be positive for other values of q outside this interval, e.g., whenever the parameter q satisfies the inequalities $(3/2) + 2k < (3-q)/(2-q) < (5/2) + 2k$,

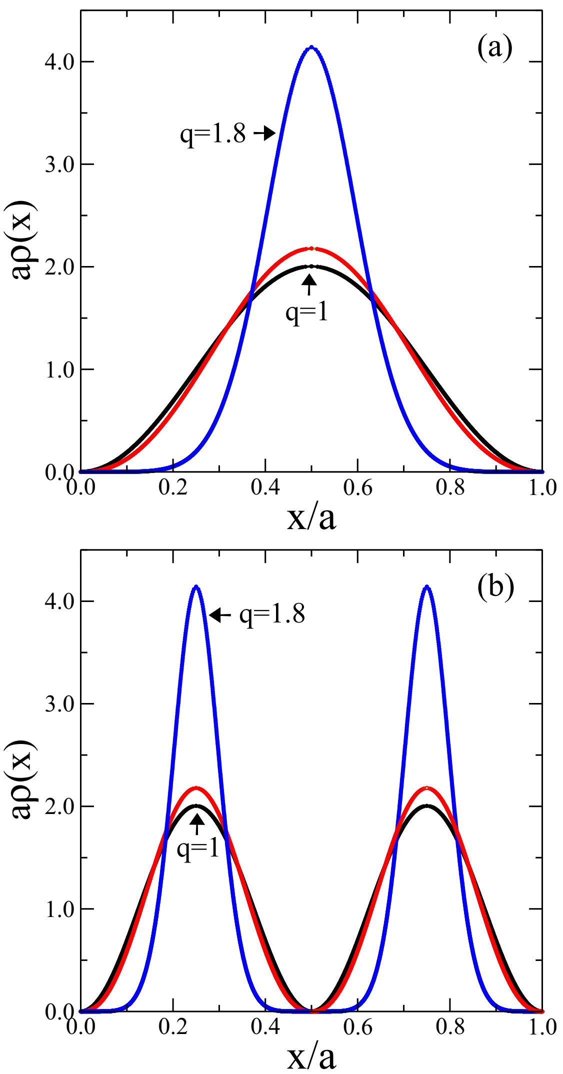

FIG. 4. (Color online) The dimensionless probability density $a\rho(x)$ [cf. Eq. (33)] for a particle in a one-dimensional infinite potential well of size a is represented for typical values of q, namely, $q = 1$, 1.25, and 1.8 (from bottom to top), in the cases $n = 1$ (a) and $n = 2$ (b). In (a) all probability densities exhibit a maximum at $(x/a) = 1/2$. In (b) all probability densities are zero at $(x/a) = 1/2$, presenting a symmetry around this point, with maxima at $(x/a) = 1/4$ and $(x/a) = 3/4$, respectively.

with k integer and $k \geqslant 1$. However, there are values of q in the range $4/3 < q < 2$ for which one obtains $\rho(x) < 0$, representing situations that deserve further analysis. Such cases may be compared with what happens to the Wigner function, which may present negative values for some values of its arguments, and so it cannot be considered as a simple probability distribution, and is often called a quasidistribution (see, e.g., Ref. [31]).

In Fig. 4 we present the dimensionless probability density $a\rho(x)$ for a particle in an infinite potential well, in the cases $n = 1$ [panel (a)] and $n = 2$ [panel (b)] and typical values of q, namely, $q = 1$, 1.25, and 1.8. For $n = 1$ one has an argument $0 \leqslant (2\tau_q x/a) \leqslant 2\tau_q$, so that $\mathrm{Sin}_q(2\tau_q x/a) \geqslant 0$ (cf. Fig. 1). From Fig. 4(a) one notices that q plays an important role for a particle with an energy $\epsilon_1(q)$, in what concerns its confinement around the central region of the well: by increasing q in the range $1 < q < 2$ the particle becomes more confined around $(x/a) = 1/2$. In this context, the present solution with an index $q > 1$ may be relevant for systems where one finds a low-energy particle localized in the central region of a confining potential. In Fig. 4(b) we show $a\rho(x)$ in the case $n = 2$ and the same values of q considered in Fig. 4(a). Now one has an argument $0 \leqslant (4\tau_q x/a) \leqslant 4\tau_q$, so that $\mathrm{Sin}_q(4\tau_q x/a)$ may yield negative values for $(x/a) > 1/2$ (cf. Fig. 1). As mentioned above, in these cases one has always real positive probabilities for $1 < q < 4/3$, as well as other values of q outside this interval (e.g., $q = 1.8$). In these cases the corresponding probability densities present a symmetry with respect to $(x/a) = 1/2$, with maxima at $(x/a) = 1/4$ and $(x/a) = 3/4$. Once again, the present solution with an index $q > 1$ may be relevant for systems where one finds a low-energy particle with the same probability for being found in two different regions, symmetrically localized around the central region of the well.

IV. UNIFORMLY ACCELERATED STATES

Let us now consider a uniformly accelerated reference frame. The corresponding spatiotemporal coordinates (x,t) are

$$t = t', \quad x = x' - \frac{1}{2}at'^2 = x' - \frac{1}{2}\frac{F}{m}t'^2, \tag{34}$$

where (x',t') are the variables associated with an inertial frame, a is the constant acceleration of the reference frame (x,t), $a = \frac{F}{m}$, and F is a constant with dimensions of force. We assume that the nonlinear field equations (12) and (13) (with zero potential, $V = 0$) hold in the inertial frame (x',t'), and also that in this frame our system is described by the q-plane-wave solution (9). Simply rewriting the q-plane-wave solution (9) (which in this section we are going to denote by $\tilde{\Psi}$ and $\tilde{\Phi}$) in terms of the new variables (x,t) does not yield a solution of the nonlinear Schrödinger equation. Similarly to what occurs with both Galilean transformations and transformations from inertial to accelerated frames applied to the linear Schrödinger equation [32], new terms are needed in the argument of the q exponential to obtain a valid solution. Let us consider the ansatz

$$\begin{aligned}\frac{\Psi}{\Psi_0} &= \left[1 - i(1-q)\left\{\omega t - k\left(x + \frac{Ft^2}{2m}\right)\right.\right.\\ &\quad \left.\left.+ \frac{F}{\hbar}\left(xt + \frac{Ft^3}{6m}\right)\right\}\right]^{1/(1-q)},\\ \frac{\Phi}{\Phi_0} &= \left[1 - i(1-q)\left\{\omega t - k\left(x + \frac{Ft^2}{2m}\right)\right.\right.\\ &\quad \left.\left.+ \frac{F}{\hbar}\left(xt + \frac{Ft^3}{6m}\right)\right\}\right]^{-q/(1-q)}.\end{aligned} \tag{35}$$

Inserting (35) in the right- and the left-hand sides of the nonlinear field equations yields

$$i\hbar\frac{\partial}{\partial t}\left(\frac{\Psi}{\Psi_0}\right)=\left[\hbar\omega-\frac{\hbar kFt}{m}+Fx+\frac{F^2t^2}{2m}\right]\left(\frac{\Psi}{\Psi_0}\right)^q, \tag{36}$$

$$-\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\Psi}{\Psi_0}\right)^{2-q}\right]=\left[\frac{\hbar^2k^2}{2m}-\frac{\hbar kFt}{m}+\frac{F^2t^2}{2m}\right]\left(\frac{\Psi}{\Psi_0}\right)^q, \tag{37}$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{\Phi}{\Phi_0}\right)=q\left[-\hbar\omega+\frac{\hbar kFt}{m}-Fx-\frac{F^2t^2}{2m}\right]\left(\frac{\Phi}{\Phi_0}\right)^{-1}, \tag{38}$$

and

$$\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\Phi}{\Phi_0}\right)^{2-q}\right]=q\left[\frac{-\hbar^2k^2}{2m}+\frac{\hbar kFt}{m}-\frac{F^2t^2}{2m}\right]\left(\frac{\Phi}{\Phi_0}\right)^{q-2}. \tag{39}$$

Substituting these results in Eqs. (12) and (13) one verifies that the ansatz (35) satisfies the nonlinear field equations, with the potential function

$$V(x)=Fx. \tag{40}$$

The nonlinear field equations with the linear potential (40) can be construed as describing the motion of a particle of mass m under the effect of a constant force $-F$ (with the concomitant potential function $V=Fx$). This is consistent with the well-known fact that the behavior of a free particle with respect to a uniformly accelerated reference frame is equivalent to the behavior of a particle in an inertial reference frame moving under the effect of a constant force. In the limit $F\to 0$, Eq. (40) reduces to the nonlinear Schrödinger equation for a free particle introduced in [4], and the solution (35) reduces to the corresponding q-plane-wave solution. Also, $q\to 1$ in Eqs. (35) corresponds to the standard linear Schrödinger-equation solution for a particle of mass m moving under a constant force $-F$.

The probability density ρ and the probability density current J associated with the accelerated q-plane-wave solutions are, respectively,

$$\rho=\frac{1}{\Omega}, \tag{41}$$

and

$$J=\frac{(1+q)}{2m\Omega}[\hbar k-Ft]. \tag{42}$$

The above form of the probability density current admits a clear physical interpretation. For a free particle we have that J is proportional to $\hbar k$ which, through the de Broglie relation, can be identified with the particle's linear momentum. For the accelerated particle J is proportional to $\hbar k_a=\hbar k-Ft$. This fact is consistent with the de Broglie relation, since it can be interpreted as describing a particle whose linear momentum is decreasing linearly with time due to the effect of a constant force $-F$.

Considering now the limit $q\to 1$ of the transformed solution (35), we verify that the original, nonaccelerated solution $\Psi_{\rm na},\Phi_{\rm na}$ and the transformed one Φ,Ψ are linked through

$$\Psi(x,t)=\exp\left[-\frac{i}{\hbar}\left(Fxt+\frac{F^2t^3}{6m}\right)\right]\Psi_{\rm na}\left(x+\frac{Ft^2}{2m},t\right),$$
$$\Phi(x,t)=\exp\left[+\frac{i}{\hbar}\left(Fxt+\frac{F^2t^3}{6m}\right)\right]\Phi_{\rm na}\left(x+\frac{Ft^2}{2m},t\right), \tag{43}$$

thus recovering the transformation rule associated with the linear Schrödinger equation [32]. Other nonlinear equations that have been discussed in the literature share with the standard linear Schrödinger equation this kind of behavior: they also "pick up" phases [as in (43)] when accelerated (or under Galilean transformations). In contrast, the nonlinear field equations analyzed here lead to extra terms appearing in the transformed solutions that are not, strictly speaking, "phases," because they cannot (except in the limit $q\to 1$) be cast as (complex) multiplicative factors of modulus 1 [as occurs in (43)].

We have obtained the transformation rule leading to additional solutions of the nonlinear field equations (12) and (13) corresponding to q plane waves "observed" from the vintage point of a uniformly accelerated frame. In the limit $q\to 1$ the transformation laws derived here coincide with those associated with time-dependent solutions of the standard, linear Schrödinger equation. The accelerated q-plane-wave solutions advanced here can be interpreted in two different ways: they can be regarded as describing a free particle "viewed" from a uniformly accelerated frame or, alternatively, as describing a particle moving under a constant force. Indeed, the nonlinear field equations governing these solutions (when expressed in terms of the accelerated frame's coordinates) incorporate an additional term involving a linear potential function $V(x)$ corresponding to a constant force. These equations confirm that, within the present field theory associated with a nonlinear generalization of the Schrödinger equation, the potential energy terms in the field equations are precisely of the form indicated in Eqs. (12) and (13). In particular, we see that the potential $V(x)$ "couples" to appropriate powers of the fields Ψ and Φ, instead of coupling just linearly to the wave function Ψ, as occurs in the case of the linear Schrödinger equation.

The time-dependent solutions of the field equations corresponding to the linear potential can be related to the family of q Gaussian solutions investigated in [24]. If we consider an ansatz for the field $\Psi(x,t)$ of the q Gaussian form

$$\Psi(x,t)=\Psi_0[1-(1-q)(\alpha x^2+\beta x+\gamma)]^{1/(1-q)}, \tag{44}$$

it is possible to verify, following a procedure similar to the one explained in [24] that the ansatz (44) indeed constitutes an exact time-dependent solution of the field equation for Ψ, provided that the complex, time-dependent parameters α, β, and γ comply with the following set of coupled, ordinary differential equations:

$$\begin{aligned} i\dot{\alpha} &= \frac{\hbar}{m}(3-q)\alpha^2, \\ i\dot{\beta} &= \frac{\hbar}{m}(3-q)\alpha\beta - \frac{F}{\hbar}, \\ i\dot{\gamma} &= \frac{\hbar}{m}\left[(1-q)\alpha\gamma - \alpha + \frac{\beta^2}{2}\right]. \end{aligned} \tag{45}$$

The above equations admit the exact solution

$$\begin{aligned} \alpha(t) &= \left[\frac{(3-q)i\hbar t}{m} + \frac{1}{\alpha_0}\right]^{-1}, \\ \beta(t) &= \left\{\frac{\beta_0}{\alpha_0} + \frac{mF}{2(3-q)\hbar^2}\left[\left(\frac{(3-q)i\hbar t}{m} + \frac{1}{\alpha_0}\right)^2 - \frac{1}{\alpha_0^2}\right]\right\} \\ &\quad \times \left[\frac{(3-q)i\hbar t}{m} + \frac{1}{\alpha_0}\right]^{-1}, \end{aligned} \tag{46}$$

where $\alpha_0 = \alpha(t=0)$ and $\beta_0 = \beta(t=0)$ are integration constants. The time dependence of the parameter $\gamma(t)$ does not admit a compact analytical expression. However, it can be expressed in terms of quadratures as

$$\begin{aligned} \gamma(t) = &\left\{\gamma_0 + \int_0^t f_0(t')\exp\left[-\int_0^{t'} f_1(t'')dt''\right]dt'\right\} \\ &\times \exp\left[\int_0^t f_1(t')dt'\right], \end{aligned} \tag{47}$$

where $\gamma_0 = \gamma(t=0)$ is an integration constant,

$$f_0(t) = \frac{i\hbar}{m}\left[\alpha - \frac{\beta^2}{2}\right], \quad f_1(t) = \frac{-i(1-q)\hbar}{m}\alpha. \tag{48}$$

and in the above expressions for f_0 and f_1, $\alpha(t)$ and $\beta(t)$ are given by (46).

The q-Gaussian-wave-packet solution (44) for the linear potential (constant force), given by the time-dependent parameters (46) and (47), is of intrinsic interest because it constitutes an additional exact time-dependent solution of the nonlinear Schrödinger equation introduced in [4] (note that the case of a linear potential was not considered in [24]). Unfortunately, when considering the nonlinear field theory advanced in [5], this q Gaussian wave function provides a solution only for the field Ψ. The evolution of the associated field Φ does not in general, for this q Gaussian Ψ wave packet, admit an analytical solution. However, in the particular case corresponding to taking the limit $\alpha_0 \to 0$, setting also $\gamma_0 = 0$, and making the identification $\beta_0 = -ik$, the q Gaussian solution reduces to the previously considered uniformly accelerated q plane wave. We see then that the uniformly accelerated q plane wave is intimately related to the q Gaussian wave packet.

The solution (35) for the coupled field equations (12) and (13) with a linear potential has a structure similar to the one exhibited by the q plane solutions advanced in [4,5], in the sense that the two fields are related by

$$\left[\frac{\Phi}{\Phi_0}\right] = \left[\frac{\Psi}{\Psi_0}\right]^{-q}. \tag{49}$$

The q-plane-wave solutions and the solutions corresponding to the linear potential are both of the form

$$\begin{aligned} \Psi &= \Psi_0[1-(1-q)iA(\vec{x},t)]^{1/(1-q)}, \\ \Phi &= \Phi_0[1-(1-q)iA(\vec{x},t)]^{-q/(1-q)}, \end{aligned} \tag{50}$$

where $A(\vec{x},t)$ is a real and universal function of $\vec{x}$ and t (universal in the sense of being independent of the parameter q). It is interesting to characterize all the solutions of the field equations that share this basic structure. Substituting the ansatz for the field Ψ in Eq. (50) into the field equation (12) leads to the following differential equation for $A(\vec{x},t)$:

$$\frac{\partial A}{\partial t} = \frac{\hbar}{2m}|\vec{\nabla}A|^2 + \frac{(1-q)\hbar}{2m}A\vec{\nabla}^2A + \frac{i\hbar}{2m}\vec{\nabla}^2A + \frac{V}{\hbar}. \tag{51}$$

If we require the function $A(\vec{x},t)$ to be real for all times and all space locations, then we need that

$$\vec{\nabla}^2A = 0. \tag{52}$$

In fact, it is plain from (51) that, if at a given initial time $t=0$ we have that $A(\vec{x},0)$ is real, then, in general, $A(\vec{x},t)$ is going to develop an imaginary part for $t>0$ due to the term $\frac{i\hbar}{2m}\vec{\nabla}^2A$ appearing in the right-hand side of (51). Therefore, the constraint (52) guarantees that an initially real function $A(\vec{x},t)$ remains real for all subsequent times. Combining Eqs. (51) and (52) leads to

$$\frac{\partial A}{\partial t} = \frac{\hbar}{2m}|\vec{\nabla}A|^2 + \frac{V}{\hbar}. \tag{53}$$

On the other hand, inserting the ansatz (50) for the field Φ into the field equation (13) leads to

$$\frac{\partial A}{\partial t} = \frac{\hbar}{2m}|\vec{\nabla}A|^2 - \frac{(1-q)\hbar}{2m}A\vec{\nabla}^2A - \frac{i\hbar}{2m}\vec{\nabla}^2A + \frac{V}{\hbar}, \tag{54}$$

which, again, under the constraint of A being real, consistently leads to the evolution equation (53).

Note that a function $A(\vec{x},t)$ evolving according the differential equation (51) and satisfying (52) at $t=0$ does not necessarily comply with this constraint at later times. In other words, the condition (52) is not invariant under the evolution determined by (51). Consequently, the equations (51) and (52) are to be regarded as two independent differential equations that have to be satisfied by a valid function $A(\vec{x},t)$ leading to legitimate solutions of the nonlinear field equations (12) and (13). It is not trivial that real solutions $A(\vec{x},t)$ satisfying both equations (51) and (52) actually exist. The q-plane-wave solutions and uniformly accelerated solutions of the field equations (12) and (13) are related to functions $A(\vec{x},t)$ that constitute explicit examples of joint solutions of Eqs. (51) and (52).

The probability density ρ and the probability density current J corresponding to solutions of the field equations having the form (50) are, respectively,

$$\rho = \frac{1}{\Omega} \tag{55}$$

and

$$J = -\frac{\hbar(1+q)}{2m\Omega}\vec{\nabla}A. \tag{56}$$

We now consider in detail the one-dimensional situation, in which the constraint

$$\vec{\nabla}^2 A = \frac{\partial^2 A}{\partial x^2} = 0 \tag{57}$$

implies that the function $A(x,t)$ has the form

$$A(x,t) = a_0(t) + a_1(t)x, \tag{58}$$

where $a_0(t)$ and $a_1(t)$ are functions of time to be determined. Equation (53) now adopts the form

$$\hbar(\dot{a}_0 + \dot{a}_1 x) = \frac{\hbar^2}{2m} a_1^2 + V(x,t). \tag{59}$$

Note that, for the sake of generality, we allow for a possible time dependence of the potential function V. It follows from the above equation that

$$\begin{aligned} \frac{\partial}{\partial x}\left[\hbar\dot{a}_0 - \frac{\hbar^2}{2m}a_1^2\right] &= \frac{\partial V(x,t)}{\partial x} - \hbar\dot{a}_1 \\ &= 0, \end{aligned} \tag{60}$$

since the quantity inside the square brackets appearing in the left-hand side of the first line of the above equation depends solely on time. Therefore, we have

$$V(x,t) = v_0(t) + v_1(t)x, \tag{61}$$

where $v_1(t) = \hbar\dot{a}_1$ and $v_0(t)$ depicts another function of time. The time-dependent functions a_0 and a_1 are related to the functions v_0 and v_1 through

$$\begin{aligned} a_0(t) = \frac{1}{\hbar}\int_0^t dt' \Bigg\{ v_0'(t) \\ + \frac{1}{2m}\left[\int_0^{t'} v_1(t'')dt'' + \hbar c_1\right]^2\Bigg\} + c_0 \end{aligned} \tag{62}$$

and

$$a_1(t) = \frac{1}{\hbar}\int_0^t v_1(t')dt' + c_1, \tag{63}$$

where $c_0 = a_0(t=0)$ and $c_1 = a_1(t=0)$ are integration constants.

V. CONCLUSIONS

We have explored some features of a recently introduced nonlinear field theory leading to a nonlinear generalized Schrödinger equation associated with the q-generalized thermostatistics. We introduced an extension of this theory incorporating potential energy terms and thus going beyond the free-particle case previously considered in the literature. Similarly to these previous studies, besides the usual field $\Psi(\vec{x},t)$, one has to introduce a second field $\Phi(\vec{x},t)$, described by an additional nonlinear field equation. On the basis of these developments we obtained additional analytical solutions of the field equations corresponding to (i) stationary solutions associated with confining potentials and (ii) time-dependent solutions for a linear potential function.

These types of solutions might be relevant for physical applications of the present nonlinear generalized Schrödinger equation. In particular, the stationary solution obtained for a particle in an infinite potential well holds for $1 \leqslant q < 2$, and it was shown that by increasing q in this interval one increases the probability for finding the particle localized around a certain position of the well. Such a behavior has some similarity with that observed in systems where one finds a low-energy particle localized inside a confining potential, e.g., a trapped atom through the interference of two or more laser fields [33], or in optical lattices [34], as well as for cold atoms and Bose-Einstein condensates, for which the NLSE of Ref. [3] has been considered (see, e.g., Refs. [35,36]).

In general, the motivations behind previous proposals for nonlinear quantum evolution equations have fallen within two main classes. On the one hand, some of these equations have been proposed as fundamental equations governing phenomena at the frontiers of our present understanding of quantum physics, particularly at the boundaries between quantum physics and gravitational physics (see, for instance, [37,38] and references therein). On the other hand, and within standard quantum physics, nonlinear-Schrödinger-like equations have been introduced as effective single-particle mean-field descriptions of complex quantum many-body systems. In the latter vein we mention the celebrated Gross-Pitaievsky equation [39]. As a final remark concerning previous applications of nonlinear Schrödinger equations, it is remarkable that instances of these equations involving a cubic nonlinearity in the wave function have also been applied in classical contexts such as the study of water waves.

With regards to the first of the above-mentioned kinds of applications, the presently discussed NLSE may be useful for describing components of dark matter. Indeed, the structure of the action variational principle leading to our NLSE suggests that it may describe particles that do not interact with the electromagnetic field [5]. Concerning the second aforementioned type of quantum applications, it is worth noticing that the NLSE exhibits an intriguing similarity with the Schrödinger equation corresponding to a particle with position- (and time-) dependent effective mass [40–43]. This equation is useful for treating (among other systems) quantum particles in nonlocal potentials (in particular, in connection with the nonlocal terms appearing in the potential associated with the energy density functional approach to the quantum many-body problem [44]). The connection between the present NLSE equation and the Schrödinger equation for particles with position-dependent mass has been recently pointed out in Refs. [42,43]. This indicates that the present NLSE may constitute an effective mean-field description, based on a single-particle wave function, of a many-body system consisting of particles interacting via nonlocal potentials. A detailed analysis of these possible applications is still premature (and certainly beyond the scope of the present work) until a more complete understanding of the properties of the NLSE and its solutions is achieved. We plan to address some of these issues in a future communication.

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