Extending SDL and LMC Complexity Measures to Quantum States

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Abstract

An extension of SDL (Shiner, Davison, Landsberg) and LMC (López-Ruiz, Mancini, Calbet) complexity measures is proposed for the quantum information context, considering that Von Neumann entropy is a natural disorder quantifier for quantum states. As a first example of application, the simple qubit was studied, presenting results similar to that obtained by applying SDL and LMC measures to a classical probability distribution. Then, for the Werner state, a mixture of Bell states, SDL and LMC measures were calculated, depending on the mixing factor γ, providing some conjectures concerning quantum systems.

Keywords: disorder; qubit; Von Neumann entropy; Werner state.

1. Introduction

Over the last two decades, several attempts to quantify complexity have been proposed with a significant amount of them using information-theoretic or computational tools to address this issue [1, 2, 3, 4]. Their use in various systems analysis justifies these efforts of complexity quantification, in order to better understand complex systems, unraveling underlying structures and sometimes bridging together very distinct systems.

To assess quantum systems there are some proposed quantum informational complexity measures [5, 6, 7, 8]; however, they are quantum extensions of the Kolmogorov's algorithmic complexity [8, 9, 10] being able to estimate necessary physical resources to implement tasks and algorithms.

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Considering the physics point of view, a perfect crystal and an isolated ideal gas, standing for totally ordered and totally disordered system, respectively, are the paradigms of simplicity. A small piece of information can describe a perfect crystal, being enough to describe its elementary cells. For the ideal gas, any accessible state has the same occurrence probability, implying maximum information [11].

Consequently, these two simple systems are on the extreme of the scales of order and information implying that a convenient definition of complexity should be proposed combining order and information [11].

Starting with the association of disorder and entropy representing thermodynamic equilibrium, for classical systems, two different complexity measures were defined: SDL (Shiner, Davison, Landsberg) and LMC (López-Ruiz, Mancini, Calbet). SDL and LMC only differ in the way of representing thermodynamic disequilibrium and are equally useful in practical cases [11, 12, 13, 14].

The idea to be presented here is to generalize SDL and LMC measures considering the Quantum Information Theory context, based on Von Neumann entropy of quantum states as a measure of disorder [15, 16]. Firstly, the concept of Von Neumann entropy will be summarized, followed by the definitions of SDL and LMC quantum complexity measures. These definitions will be applied to two relevant examples: the single qubit and the Werner state, closing with some conclusions about quantum complexity measures.

2. Quantum SDL and Quantum LMC complexity measures

From now on, H is considered to be a finite dimension(N) Hilbert space containing the states $|\Psi>$ of a quantum system. To each state $|\Psi>$ corresponds a density matrix ρ, according to the usual notation of Quantum Mechanics [16, 17].

Under these conditions, the Von Neumann (VN) entropy is defined as:

$$S(\rho) = tr[\rho\, log_2(\rho)], \tag{1}$$

with tr representing matrix trace. Matrix $log_2(\rho)$ is obtained by calculating the logarithm of each element of ρ and, when the element is zero, the corresponding element of $log_2(\rho)$ is considered to be zero, as usual even in the classical information theory [16].

Considering $\lambda_i, i = 1, ..., N$ the eigenvalues of ρ, the VN entropy can be rewritten as:

$$S(\rho) = \sum_{i=1}^{N} \lambda_i\, log_2(\lambda_i). \tag{2}$$

Expression (2) of Von Neumann entropy is verified to be analogous to the formula to calculate Shannon entropy for a discrete distribution with probability values given by the set of the eigenvalues λ_i.

Consequently, the maximum possible value for $S(\rho)$ corresponds to the equiprobable distribution, i.e., the distribution with all λ_i equal to $1/N$. Therefore:

$$S_{max} = log_2 N. \tag{3}$$

2.1. Quantum SDL complexity measure

Analogously to the classical case, quantum SDL complexity measure is defined by considering that quantum disorder (Δ_q) of a state can be expressed by the relation between its VN entropy and the maximum possible value of the VN entropy:

$$\Delta_q(\rho) = \frac{S(\rho)}{S_{max}}. \tag{4}$$

As SDL classical complexity measure, quantum complexity measure is defined by the weighted product of the quantum disorder (Δ_q) by the quantum order ($1 - \Delta_q$) [12]:

$$(SDL)_q = (\Delta_q)^\alpha \, (1 - \Delta_q)^\beta, \tag{5}$$

with α and β representing how order and disorder are weighted. The natural choice for these parameters is $\alpha = \beta = 1$, providing equal contributions of the two factors to complexity. Under this hypothesis:

$$(SDL)_q = (\Delta_q) \, (1 - \Delta_q). \tag{6}$$

2.2. Quantum LMC complexity measure

LMC complexity measure differs from SDL, given by (6), by the replacement of the disorder term ($1 - \Delta_q$) by a term called disequilibrium D_q that measures the distance between the eigenvalues probability distribution and the equiprobable one [11], and is defined as:

$$D_q = \sum_1^N (\lambda_i - 1/N)^2. \tag{7}$$

Then, quantum LMC complexity measure is given by:

$$(LMC)_q = \Delta_q \, D_q \tag{8}$$

3. Examples

In this section, two problems are studied: how quantum complexity of a single qubit depends on the probability distribution between the states $|0>$ and $|1>$ and how quantum complexity of a Werner state is related to the mixing factor.

3.1. Complexity of a single q-bit

In order to calculate the complexity of a single qubit, it will be represented by:

$$|q>= a|0> +b|1>, \tag{9}$$

with $|0>$ and $|1>$ being pure states in a two dimension Hilbert space and $p = |a|^2$ and $1 - p = |b|^2$, with $0 \leq p \leq 1$.

Calculating the corresponding density matrix:

$$\rho = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}.$$

Considering the expression of ρ, the VN entropy, the $(SDL)_q$ and the $(LMC)_q$ were calculated as p is varied and the results are shown in Fig. 1.

3.2. Complexity of a mixing entangled state

The property of entanglement in quantum states can be expressed in terms of qubits by the Bell states, defined in a four-dimension Hilbert space as [18]:

$$|\Phi^+>= \frac{1}{\sqrt{2}}[|00> +|11>]; \quad |\Phi^->= \frac{1}{\sqrt{2}}[|00> -|11>];$$
$$|\Psi^+>= \frac{1}{\sqrt{2}}[|01> +|10>]; \quad |\Psi^->= \frac{1}{\sqrt{2}}[|01> -|10>].$$

The Werner state is an emblematic example showing that, sometimes, an entangled mixed state does not violate Bell's inequalities [18]. Here it will be used for another objective: show how the mixing factor affects complexity calculation in case of entanglement.

The Werner state version to be considered here has its density matrix given by:

$$\rho_w = \gamma[|\Psi^- >< \Psi^-|] + \frac{1-\gamma}{3}[|\Psi^+ >< \Psi^+| + |\Phi^+ >< \Phi^+| + |\Phi^- >< \Phi^-|], \tag{10}$$

with $\gamma \in [0, 1]$ representing the mixing degree.

Expressing (10) in a matrix form in the four dimension double-qubit basis:

$$\rho_w = \begin{pmatrix} \frac{1-\gamma}{3} & 0 & 0 & 0 \\ 0 & \frac{1+2\gamma}{6} & \frac{1-4\gamma}{6} & 0 \\ 0 & \frac{1-4\gamma}{6} & \frac{1+2\gamma}{6} & 0 \\ 0 & 0 & 0 & \frac{1-\gamma}{3} \end{pmatrix}.$$

By using the definitions of section 2, SDL and LMC quantum complexity measures were calculated for the Werner state given by density matrix ρ_w, varying mixing degree γ. The results are shown in Fig. 2.

4. Conclusions

When extending the concepts of classical SDL and LMC complexity measures to the quantum information world by considering VN entropy a disorder measure, some consequences appear:

- the results for the single qubit systems (Fig. 1) were compatible with those presented in [11, 12] as the qubit VN entropy reproduces the Shannon entropy corresponding to a discrete probability distribution;
- for the single qubit, $(SDL)_q$ and $(LMC)_q$ present qualitatively the same results (Fig. 1);
- for the mixed entangled state (Werner state) the results are robust (Fig. 2), too. Both complexity measures resulted zero for the conditions of $S = 1$ or $S = 0$, corresponding to maximum disorder and maximum order, respectively;
- for the Werner state, the maximum complexity value corresponds to a mixing factor of 0.8 for both measures (Fig. 2;
- concerning the Werner state $(SDL)_q$ and $(LMC)_q$ present qualitatively the same results (Fig. 2).

Based on those facts, it can be concluded that either $(SDL)_q$ or $(LMC)_q$ are good alternatives to measure the complexity of quantum systems.

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Figure captions

Figure 1. Entropy and quantum complexities for a single qubit.

Figure 2. Entropy and quantum complexities for the Werner state varying the mixing degree.