Nonlinear Relativistic and Quantum Equations with a Common Type of Solution

F. D. Nobre,[1,*] M. A. Rego-Monteiro,[1] and C. Tsallis[1,2]

[1]*Centro Brasileiro de Pesquisas Físicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro–RJ Brazil*

[2]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*

(Received 25 October 2010; published 4 April 2011)

Generalizations of the three main equations of quantum physics, namely, the Schrödinger, Klein-Gordon, and Dirac equations, are proposed. Nonlinear terms, characterized by exponents depending on an index q, are considered in such a way that the standard linear equations are recovered in the limit $q \to 1$. Interestingly, these equations present a common, solitonlike, traveling solution, which is written in terms of the q-exponential function that naturally emerges within nonextensive statistical mechanics. In all cases, the well-known Einstein energy-momentum relation is preserved for arbitrary values of q.

DOI: 10.1103/PhysRevLett.106.140601 PACS numbers: 05.90.+m, 02.30.Jr, 03.65.Pm, 05.45.Yv

The applicability of linear equations in physics is usually restricted to idealized systems, being valid for media characterized by specific conditions, such as homogeneity, isotropy, and translational invariance, with particles interacting through short-range forces and with a dynamical behavior characterized by short-time memory. However, many real systems—specially the ones within the realm of complex systems—do not fulfill these requirements, and usually exhibit complicated collective behavior associated to NL phenomena [1]. Accordingly, the study of nonlinear equations (NLEs) has opened a new area of physics, attracting a lot of interest due to the possibility of describing many real situations. Since finding analytical solutions of NLEs can be a hard task, particularly in the case of NL differential equations [2], very frequently one has to make use of numerical procedures and so, a considerable advance has been attained lately.

Among the most studied NL differential equations we have the sine-Gordon [3], the NL Schrödinger [4], and the Fokker-Planck ones [5]. In these cases, the NL contributions lead to important novel insights, relevant for modeling several physical new aspects. Two special types of solutions have generated a lot of interest, particularly in the sine-Gordon and NL Schrödinger equations; these are known as solitons (or solitary waves) and breathers [1–4]. In both cases one has compact traveling solutions, the first one being characterized by a spatial part that does not deform throughout the evolution, whereas the later presents an oscillating width as propagation occurs. An important property of these two solutions concerns their square integrability, allowing for an appropriate normalization. Because of the modulation of the wave function, these solutions are considered to be relevant in diverse areas of physics, including nonlinear optics, superconductivity, plasma physics, and deep water waves [1].

In the most common NL Schrödinger equation, one introduces a new cubic term in the wave function (see, for instance, [4]) which, for some particular type of solution, is responsible for the modulation of the wave function. Such an addition of extra NL terms has been much used in the literature for constructing NLEs. A different approach consists in transforming one or more linear terms into NL ones, as usually happens in NL Fokker-Planck equations [5]. In this case, whereas the linear Fokker-Planck equation is associated to normal diffusion and to the Boltzmann-Gibbs entropy, its NL counterparts are usually related to anomalous-diffusion phenomena [6] and to generalized entropies [7], like the nonadditive one which yields nonextensive statistical mechanics [8–10].

This theory emerged from a generalization of the Boltzmann-Gibbs entropy, through the introduction of an index q ($q \to 1$ recovers the standard case). Since then, considerable progress has been achieved, leading to generalized functions, distributions, important equations of physics, and even new forms of the central limit theorem [11]. In particular, the q-Gaussian distribution, which generalizes the standard Gaussian, appears naturally by extremizing the entropy [8], or from the solution of the corresponding nonlinear Fokker-Planck equation [12], and has been very useful for experiments in many real systems [9,10]. Among many others, we may mention the following: (i) the velocities of cold atoms in dissipative optical lattices [13]; (ii) the velocities of particles in quasi-two-dimensional dusty plasma [14]; (iii) single ions in radio frequency traps interacting with a classical buffer gas [15]; (iv) the relaxation curves of RKKY spin glasses, like CuMn and AuFe [16]; (v) transverse momenta distributions at LHC experiments [17].

Herein we introduce NL generalizations of the three main equations of quantum physics, namely, the Schrödinger, Klein-Gordon, and Dirac equations. The present proposals consist in extending linear terms into NL ones, as done for the NL Fokker-Planck equations. An interesting aspect about these generalizations is that they may be formulated easily in arbitrary dimensions, whose exact solutions are here presented. These solutions are all expressed in terms of the q-exponential function $\exp_q(u)$ that naturally emerges in nonextensive statistics [9,10]. For a pure imaginary iu, one defines $\exp_q(iu)$ as the principal value of

$$\exp_q(iu) = [1 + (1-q)iu]^{1/1-q}; \quad \exp_1(iu) \equiv \exp(iu). \tag{1}$$

The above function satisfies [18],

$$\exp_q(\pm iu) = \cos_q(u) \pm i\sin_q(u), \tag{2}$$

$$\cos_q(u) = \rho_q(u)\cos\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}, \tag{3}$$

$$\sin_q(u) = \rho_q(u)\sin\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}, \tag{4}$$

$$\rho_q(u) = [1 + (1-q)^2u^2]^{1/[2(1-q)]}, \tag{5}$$

$$\exp_q(iu)\exp_q(-iu) = \cos_q^2(u) + \sin_q^2(u) = \rho_q^2(u). \tag{6}$$

Notice that $\exp_q[i(u_1 + u_2)] \neq \exp_q(iu_1)\exp_q(iu_2)$ for $q \neq 1$ [9]. As a consequence of Eqs. (2)–(6), the q exponential of a pure imaginary presents an oscillatory behavior with a varying amplitude $\rho_q(u)$ that decreases (increases) for $1 < q < 3$ ($q < 1$). By integrating Eq. (6) from $-\infty$ to $+\infty$, we verify the physically important property of square integrability for $1 < q < 3$, whereas such integral diverges in both limits $q \to 1$ and $q \to 3$ and for $q < 1$ [19]. More precisely, $\exp_q(iu)$ is modulated by the q Gaussian in Eq. (5). This is the type of solution that we focus on throughout the present Letter.

Let us first consider the simple one-dimensional linear wave equation

$$\frac{\partial^2\Phi(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\Phi(x,t)}{\partial t^2}, \tag{7}$$

for which any function of the type $\Phi(kx - \omega t)$, twice differentiable, is a solution. In particular, one may have a q-plane wave [19],

$$\Phi(x,t) = \Phi_0\exp_q[i(kx - \omega t)]; \qquad [\Phi_0 \equiv \Phi(0,0)], \tag{8}$$

as a solution of the equation above, provided that $\omega = ck$. Let us remind the reader that the above q-plane wave cannot be decomposed, for $q \neq 1$, into spatial and temporal factors, as it happens in many solutions of NLEs [2]. Moreover, the amplitude of the q-plane wave for $1 < q < 3$ decreases when its argument $(kx - \omega t)$ increases. Because of this property, this new type of solution may well be applicable to various nonlinear physical phenomena. Let us stress that, in the particular situation where $x = ct$, one has $\Phi(x,t) = \Phi_0 (\forall\ t)$, consequently the q-plane wave behaves like a soliton propagating with a velocity $c = \omega/k$. This enables the approach of nonlinear excitations which do not deform in time and should be relevant, e.g., in nonlinear optics and plasma physics.

Now, since Eq. (7) is a linear equation, the following linear combination,

$$\Phi(x,t) = \sum_j \Phi_{0j}\exp_{q_j}[i(kx - \omega t)], \tag{9}$$

is also a solution with the same dispersion relation. A given physical system may be characterized by a single value of q. If $1 < q < 3$, we have $\Phi(\pm\infty, t) = 0 (\forall\ t)$; by specifying the rate of decay of the q-plane wave amplitude (which should be a characteristic of a given physical system), i.e., its modulation, Eq. (5), we may determine the appropriate value of q.

Another important aspect of the q-plane wave concerns its immediate extension to d dimensions,

$$\Phi(\vec{x},t) = \Phi_0\exp_q[i(\vec{k}\cdot\vec{x} - \omega t)], \tag{10}$$

which, due to the well-known properties of the scalar product $\vec{k}\cdot\vec{x} = \sum_{n=1}^d k_n x_n$, exhibits invariance under rotations. If we take into account that $d\exp_q(z)/dz = [\exp_q(z)]^q$ and $d^2\exp_q(z)/dz^2 = q[\exp_q(z)]^{2q-1}$ we obtain, for the d-dimensional Laplacian,

$$\nabla^2\Phi(\vec{x},t) = -q\left(\sum_{n=1}^d k_n^2\right)\Phi_0\{\exp_q[i(\vec{k}\cdot\vec{x} - \omega t)]\}^{2q-1}. \tag{11}$$

From the results above one sees that the d-dimensional q-plane wave of Eq. (10) satisfies the linear wave equation,

$$\nabla^2\Phi(\vec{x},t) = \frac{1}{c^2}\frac{\partial^2\Phi(\vec{x},t)}{\partial t^2} \quad \Rightarrow \quad \omega = ck, \tag{12}$$

with $k = \sqrt{\sum_{n=1}^d k_n^2}$, showing that the resulting dispersion relation is q invariant.

We will now focus on NLEs for which the above d-dimensional q-plane wave is an exact solution. It should be emphasized at this point that many NLEs in the literature are usually formulated in one dimension and that their extension to d dimensions may not be always an easy task. Accordingly, let us introduce the following d-dimensional NL generalization of the Schrödinger equation for a particle of mass m,

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{2-q}. \tag{13}$$

We notice that the scaling of the wave function by Φ_0 guarantees the correct physical dimensionalities for all terms. This scaling becomes irrelevant only for linear equations [e.g., in the particular case $q = 1$ of Eq. (13)]. Consistently, the energy and momentum operators are generalized as $\hat{E} = i\hbar D_t$ and $\hat{p}_n = -i\hbar D_{x_n}$ respectively, where $D_u f(u) \equiv [f(u)]^{1-q} df(u)/du$. These operators, when acting on the q-exponential $\exp_q[i(\vec{k}\cdot\vec{x} - \omega t)]$, yield the energy $E = \hbar\omega$ and momentum $\vec{p} = \hbar\vec{k}$. Now, if one considers the q-plane wave solution of Eq. (10) by using $\vec{k} \to \vec{p}/\hbar$ and $\omega \to E/\hbar$, one verifies that this new form is a solution of the equation above, with $E = p^2/2m$, for all values of q. The NL Schrödinger equation of Eq. (13) shows the same structure of the NL Fokker-Planck equation of Refs. [12] in the absence of an external potential, which appears in nonextensive statistical mechanics [8,9]. Essentially, it represents the analogue of the porous-medium equation [20], very common in anomalous-diffusion phenomena [6], but with an imaginary time. One well-known solution of the porous-medium equation is due to Barenblatt [21]. This solution presents a structure very similar to a q Gaussian, written in terms of a power law like the one in Eq. (8).

However, Eq. (13) differs from previous formulations [4], where one adds a new NL term (in most cases, a cubic nonlinearity in the wave function) to the two existing linear terms. This extra NL term is responsible for the modulation of the wave function, which has made such equations relevant for many physical phenomena since some decades. Particularly, the breatherlike solutions are characterized by a time dependence of the linear type, i.e., $e^{i\omega t}$, such that their nonlinearities are essentially manifested through their spatial dependences. In its discrete formulation, this equation leads to discrete breathers, much studied in the literature due to their potential applications in lattice dynamics of solids, coupled arrays of Josephson junctions, and localization of waves in random lattices, among others [22,23]. The main differences between Eq. (13) and the existing forms of NL Schrödinger equations in the literature are the following: (i) instead of adding an extra term in which the nonlinearity is introduced, we modify the spatial second-derivative term; (ii) the equation, together with the proposed solution, are easily extended from one to d dimensions; (iii) the corresponding solution of Eq. (13) manifests nonlinearity in both space and time, through a modulation in these two variables, which keeps the norm finite for all $(\vec{x}, t)$; (iv) the well-known energy spectrum $E = p^2/2m$ is preserved for all q. Therefore Eq. (13), together with the simple solution in Eq. (10), emerge as possible descriptions of physical phenomena, like the propagation of nonlinear pulses in optical fibers for carrying information ("bits").

Another category of NL equations studied in the literature concerns the class of NL Klein-Gordon equations (see, for instance, [2,3,23–25]). As in the previous case, in most NL Klein-Gordon equations the second-derivative terms are left unchanged and the nonlinearity is introduced by means of extra terms containing powers of the Klein-Gordon field [2], e.g., cubic [24,25] or quartic [23] terms. If one extends this nonlinearity to a general functional of the field, then the sine-Gordon equation [3], which contains a term expressed as a sine of the field, may be also included in this category [1,22]. Several types of solutions have been proposed in the literature for such NL Klein-Gordon equations, particularly those of the Barenblatt kind [21] and breatherlike ones [22,23]. In other cases the solutions are characterized by an amplitude that may decay in both space and time, multiplied by a periodically oscillating part in the form $e^{i(Ax+Bt)}$ (A and B constants) [24,25]. The solution presented in [24,25] is qualitatively similar to the one-dimensional q-plane wave. However, it may not be trivially extended to d dimensions.

Herein we propose a new NL Klein-Gordon equation in d dimensions, namely

$$\nabla^2\Phi(\vec{x}, t) = \frac{1}{c^2}\frac{\partial^2\Phi(\vec{x}, t)}{\partial t^2} + q\frac{m^2c^2}{\hbar^2}\Phi(\vec{x}, t)\left[\frac{\Phi(\vec{x}, t)}{\Phi_0}\right]^{2(q-1)}. \tag{14}$$

One may verify easily that the same q-plane wave used for the NL Schrödinger equation is a solution of Eq. (14), preserving for all q the Einstein relation

$$E^2 = p^2c^2 + m^2c^4. \tag{15}$$

It should be mentioned that, to our knowledge, the nonlinear term in Eq. (14) is new and different from those of previous formulations, even in the particular cases $q = 2$ and $q = 5/2$, where one gets cubic and quartic terms in the field, respectively. In previous works, the nonlinear term is constructed by multiplying the wave function by a power of its modulus, leading to different types of solutions and different energy spectra. Another important aspect of Eq. (14) concerns its Lorentz invariance: since this property is directly related to the first two terms (containing derivatives) [26], which have not been changed herein, then Eq. (14) remains invariant under Lorentz transformation. Some previous NL Klein-Gordon equations are not invariant under usual Lorentz transformation [24,25] and require its generalization [25].

Along the same line we introduce a NL form for the Dirac equation. The present proposal represents, to the best of our knowledge, a new one (see Ref. [27] for a previous, different formulation). In this case, we will restrict ourselves to $d = 3$ spatial dimensions; let us then introduce the following generalized Dirac equation,

$$i\hbar\frac{\partial\Phi(\vec{x}, t)}{\partial t} + i\hbar c(\vec{\alpha}\cdot\vec{\nabla})\Phi(\vec{x}, t) = \beta mc^2A^{(q)}(\vec{x}, t)\Phi(\vec{x}, t), \tag{16}$$

where α_x, α_y, α_z (written in terms of the Pauli spin matrices) and β (written in terms of the 2×2 identity matrix I) are the standard 4×4 matrices [26]. The new, q-dependent, term is given by the 4×4 diagonal matrix $A^{(q)}_{ij}(\vec{x}, t) = \delta_{ij}[\Phi_j(\vec{x}, t)/a_j]^{q-1}$, where $\{a_j\}$ are complex constants ($A^{(1)}_{ij}(\vec{x}, t) = \delta_{ij}$). The solution of Eq. (16) we focus on is the following four-component column matrix

$$\Phi(\vec{x}, t) \equiv \begin{pmatrix}\Phi_1(\vec{x}, t)\\ \Phi_2(\vec{x}, t)\\ \Phi_3(\vec{x}, t)\\ \Phi_4(\vec{x}, t)\end{pmatrix} = \begin{pmatrix}a_1\\ a_2\\ a_3\\ a_4\end{pmatrix}\exp_q\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)\right]. \tag{17}$$

Substituting this four-component vector into Eq. (16), we get, for the coefficients $\{a_j\}$, precisely the same set of four algebraic equations corresponding to the linear case [see p. 803, Eq. (15.45b) of [26]]. These equations have, for all q, a nontrivial solution only if the Einstein energy-momentum relation [Eq. (15)] is satisfied. The proposal in Eq. (16) differs from that in [27] in the same sense of the herein proposed NL Schrödinger and Klein-Gordon equations; i.e., the nonlinearity is introduced by generalizing an existing term, rather than adding an extra NL term. As a consequence, our solutions are given by the above-mentioned q exponential, thus preserving relation (15), in contrast to those in Ref. [27], which are typically written in terms of the standard exponential. The positive and negative parts of the present energy spectrum are naturally expected to be, respectively, associated to a particle and its corresponding antiparticle. Contrary to the standard plane waves (case $q = 1$), one has q-plane wave solutions, which are square integrable, i.e., with a finite norm for $1 < q < 3$, representing localized particles.

To conclude, we have shown that the q-plane waves, which consist of a generalization—within the framework of nonextensive statistical mechanics—of the standard plane waves, represent a very general type of solutions for some physical nonlinear equations. Besides the d-dimensional linear wave equation, it appears also as a solution of important nonlinear generalizations of the main equations of quantum physics, namely, nonlinear d-dimensional Schrödinger and Klein-Gordon equations, and a three-dimensional Dirac equation (its generalization to d dimensions should in principle be straightforward; see [28] for the linear case). In contrast to other generalizations known in the literature, where nonlinear terms are added to the usual linear ones—and whose extensions from one- to d-dimensional space are in many cases not particularly straightforward—the present proposals may be easily formulated in d dimensions. They consist in fact in modifying existing terms by introducing powers characterized by an index q. The standard linear equations are recovered in the limit $q \to 1$. Furthermore, the equations and solutions presented here preserve, in all cases, the usual energy-momentum relations. Because of its simplicity and properties described herein, this type of solution represents a good candidate for describing several nonlinear physical phenomena characterized by oscillatory motion with modulation in both space and time, like those appearing in superconductivity, plasma physics, nonlinear optics, and lattice dynamics of solids. Naturally, at the present early stage, specific applications of this theory are still elusive.

We thank Professor E. M. F. Curado for fruitful conversations. The partial financial support from CNPq and FAPERJ (Brazilian agencies) is acknowledged.

*Corresponding author.
fdnobre@cbpf.br

[1] A. C. Scott, *Encyclopedia of Nonlinear Science* (Taylor and Francis, New York, 2005); *The Nonlinear Universe* (Springer, Berlin, 2007).
[2] A. D. Polyanin and V. F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations* (Chapman and Hall/CRC, Boca Raton, 2004).
[3] Y. Frenkel and T. Kontorova, J. Phys. USSR **1**, 137 (1939).
[4] C. Sulem and P.-L. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse* (Springer, New York, 1999).
[5] T. D. Frank, *Nonlinear Fokker-Planck Equations: Fundamentals and Applications* (Springer, Berlin, 2005).
[6] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[7] V. Schwämmle, F. D. Nobre, and E. M. F. Curado, Phys. Rev. E **76**, 041123 (2007); V. Schwämmle, E. M. F. Curado, and F. D. Nobre, Eur. Phys. J. B **58**, 159 (2007).
[8] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[9] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics* (Springer, New York, 2009).
[10] C. Tsallis, Braz. J. Phys. **39**, 337 (2009).
[11] S. Umarov, C. Tsallis, and S. Steinberg, Milan J. Math. **76**, 307 (2008); S. Umarov, C. Tsallis, M. Gell-Mann, and S. Steinberg, J. Math. Phys. (N.Y.) **51**, 033502 (2010).
[12] A. R. Plastino and A. Plastino, Physica (Amsterdam) **222A**, 347 (1995); C. Tsallis and D. J. Bukman, Phys. Rev. E **54**, R2197 (1996).
[13] P. Douglas, S. Bergamini, and F. Renzoni, Phys. Rev. Lett. **96**, 110601 (2006).
[14] B. Liu and J. Goree, Phys. Rev. Lett. **100**, 055003 (2008).
[15] R. G. DeVoe, Phys. Rev. Lett. **102**, 063001 (2009).
[16] R. M. Pickup, R. Cywinski, C. Pappas, B. Farago, and P. Fouquet, Phys. Rev. Lett. **102**, 097202 (2009).
[17] V. Khachatryan *et al.* (CMS Collaboration), Phys. Rev. Lett. **105**, 022002 (2010).
[18] E. P. Borges, J. Phys. A **31**, 5281 (1998).
[19] M. Jauregui and C. Tsallis, J. Math. Phys. (N.Y.) **51**, 063304 (2010); A. Chevreuil, A. Plastino, and C. Vignat, J. Math. Phys. (N.Y.) **51**, 093502 (2010).
[20] J. L. Vázquez, *The Porous Medium Equation* (Oxford University Press, Oxford, 2007).
[21] G. I. Barenblatt and Ya. B. Zel'dovich, Annu. Rev. Fluid Mech. **4**, 285 (1972).
[22] S. Flach and C. R. Willis, Phys. Rep. **295**, 181 (1998).
[23] G. Kopidakis, S. Aubry, and G. P. Tsironis, Phys. Rev. Lett. **87**, 165501 (2001); G. Kopidakis, S. Komineas, S. Flach, and S. Aubry, Phys. Rev. Lett. **100**, 084103 (2008).
[24] F. T. Hioe, J. Phys. A **36**, 7307 (2003).
[25] F. T. Hioe, J. Phys. A **37**, 173 (2004).
[26] R. L. Liboff, *Introductory Quantum Mechanics* (Addison Wesley, San Francisco, 2003), 4th ed.
[27] W. I. Fushchich and W. M. Shtelen, J. Phys. A **16**, 271 (1983).
[28] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison Wesley, Reading, MA, 2005).