

# Generalized Nonlinear Proca Equation and Its Free-Particle Solutions

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We introduce a non-linear extension of Proca's field theory for massive vector (spin 1) bosons. The associated relativistic nonlinear wave equation is related to recently advanced nonlinear extensions of the Schroedinger, Dirac, and Klein-Gordon equations inspired on the non-extensive generalized thermostatics. This is a theoretical framework that has been applied in recent years to several problems in nuclear and particle physics, gravitational physics, and quantum field theory. The nonlinear Proca equation investigated here has a power-law nonlinearity characterized by a real parameter  $q$  (formally corresponding to the Tsallis entropic parameter) in such a way that the standard linear Proca wave equation is recovered in the limit  $q \rightarrow 1$ . We derive the nonlinear Proca equation from a Lagrangian that, besides the usual vectorial field  $\Psi^\mu(\vec{x}, t)$ , involves an additional field  $\Phi^\mu(\vec{x}, t)$ . We obtain exact time dependent soliton-like solutions for these fields having the form of a  $q$ -plane wave, and show that both field equations lead to the relativistic energy-momentum relation  $E^2 = p^2c^2 + m^2c^4$  for all values of  $q$ . This suggests that the present nonlinear theory constitutes a new field theoretical representation of particle dynamics. In the limit of massless particles the present  $q$ -generalized Proca theory reduces to Maxwell electromagnetism, and the  $q$ -plane waves yield localized, transverse solutions of Maxwell equations. Physical consequences and possible applications are discussed.

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## I. INTRODUCTION

The aim of the present contribution is to advance and explore some features of a nonlinear extension of Proca's field equation. This proposal is motivated by recent developments concerning nonlinear extensions of the Schroedinger, Dirac, and Klein-Gordon equations [1–7] related to the non-extensive generalized thermostatistics [8, 9]. These nonlinear equations are closely related to a family of power-law nonlinear Fokker-Planck equations that describe the spatio-temporal behavior of various physical systems and processes and have been studied intensively in recent years [10–17].

The Proca equation [18] constitutes, along with the Dirac and the Klein-Gordon equations, one of the fundamental relativistic wave equations [19]. It describes massive vector (spin 1) boson fields. Historically it played an important role due, among other things, to its relation with Yukawa's work on mesons [20]. Proca's equation shields a generalization of Maxwell's electromagnetic field theory, by incorporating the effects of a finite rest mass for the photon [21]. As such, it is a basic ingredient of the theoretical framework for experimental studies aiming at the determination of upper bounds for the mass of the photon [22, 23]. This is a basic line of enquiry that can be regarded as having its origins in experimental work by Cavendish and Maxwell, with their explorations of possible deviations from Coulomb's electrostatic force law [22] (although they, obviously, did not formulate this problem in terms of the mass of the photon). Within the modern approach, the Proca field equation allows for definite quantitative predictions concerning diverse physical effects originating in a finite mass of the photon (including the aforementioned deviations from Coulomb's law). This, in turn, motivates concrete experiments for the search of the mentioned effects, and for determining concomitant upper bounds for the photon's rest mass. Besides the problem of establishing bounds to the photon mass, the study of the Proca field equation has been a subject of constant interest in theoretical physics [24–29]. Proca-like modifications of electromagnetism have been considered in order to explore possible violations of Lorentz

invariance at large distances [25]. The Proca equation with a negative square-mass constitutes a theoretical tool for the analysis of tachyon physics [26]. Among other interesting aspects of the Proca field we can mention the rich variety of phenomena associated with the coupled Einstein-Proca field equations [28], which constitute a natural extension of the celebrated Einstein-Maxwell equations. Last, but certainly not least, the Proca field may be related to dark matter [29], whose nature is one of the most pressing open problems in contemporary Science.

The nonlinear Schroedinger, Dirac, and Klein-Gordon equations investigated in [1–7] share the physically appealing property of admitting (in the case of vanishing interactions) exact soliton-like localized solutions that behave as free particles, in the sense of complying with the celebrated Einstein-Planck-de Broglie relations connecting frequency and wave number respectively, with energy and momentum. Given these previous developments, it is natural to ask if the corresponding nonlinear extension can also be implemented for massive vector bosons. This is the question we are going to explore in the present contribution. Of especial relevance for our purposes is the nonlinear Klein-Gordon equation proposed in [1]. It exhibits a nonlinearity in the mass term which is proportional to a power of the wave function  $\Phi(x, t)$ . For the above mentioned exact solutions the wave function  $\Phi(x, t)$  depends on space and time only through the combination  $x - vt$ . This space-time dependence corresponds to a uniform translation at a constant velocity  $v$  without change in the shape of the wave function. These soliton-like solutions are called  $q$ -plane waves and, as already mentioned, are compatible with the Einstein-Planck-de Broglie relations  $E = \hbar\omega$  and  $p = \hbar k$ , satisfying the relativistic energy-momentum relation  $E^2 = c^2p^2 + m^2c^4$ . The  $q$ -plane waves constitute a generalization of the standard exponential plane waves that arise naturally within a theoretical framework that extends the Boltzmann-Gibbs (BG) entropy and statistical mechanics on the basis of a power-law entropic functional  $S_q$ . This functional is parameterized by a real index  $q$ , the usual BG formalism being recovered in the limit  $q \rightarrow 1$ . The  $q$ -plane waves are complex-valued versions of the  $q$ -exponential distributions that optimize the  $S_q$  entropies under appropriate constraints. These distributions are at the core of the alluded extension of the BG thermostatics, which has been applied in recent years to a variegated set of physical scenarios. In particular, several applications to problems in nuclear and particle physics, as well as in quantum field theory, have been recently advanced. As examples we can mention applications of the  $q$ -nonextensive thermostatics to the study of the nu-

clear equation of state [30–32], to neutron stars [33, 34], to the thermodynamics of hadron systems [35–38], to proton-proton and heavy ion collisions [39–45], to quantum chromodynamics [46–48], to cosmic rays [49], to the Thomas-Fermi model of self-gravitating systems [50], to fractal deformations of quantum statistics [51], and to the entropic-force approach to gravitation [52, 53]. Intriguing connections between the non extensive thermostatistical formalism and  $q$ -deformed dynamics have been suggested [54]. The  $q$ -nonextensive thermostatsitics has also stimulated the exploration of other non-additive entropic functionals that have been applied, for instance, to black hole thermodynamics [55, 56].

The paper is organized as follows. In Section II we introduce a Lagrangian for the generalized Proca field that leads to the nonlinear Proca field equations. We study some of its main properties, with special emphasis on the  $q$ -plane wave soliton-like solutions. In Section III we consider the limit of massless particles, obtaining a new family of wave-packet solutions of Maxwell equations. Some conclusions and final remarks are given in Section IV.

## II. LAGRANGIAN APPROACH FOR NONLINEAR PROCA EQUATIONS

Let us introduce the four-dimensional space-time operators [19],

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \equiv \left\{ \frac{\partial}{\partial(ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right\} ; \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \equiv \left\{ \frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} , \quad (1)$$

as well as the contravariant and covariant vectors

$$\Psi^\mu \equiv (\Psi_0, \Psi_x, \Psi_y, \Psi_z); \quad \Psi_\mu \equiv (\Psi_0, -\Psi_x, -\Psi_y, -\Psi_z), \quad (2)$$

$$\Phi^\mu \equiv (\Phi_0, \Phi_x, \Phi_y, \Phi_z); \quad \Phi_\mu \equiv (\Phi_0, -\Phi_x, -\Phi_y, -\Phi_z). \quad (3)$$

As it will be shown below, these two vector fields are necessary for a consistent field theory, similarly to the recent nonlinear versions of the Schroedinger and Klein-Gordon equations [2, 7]. For this, we introduce the following Lagrangian density,

$$\mathcal{L} = A \left\{ -\frac{1}{2} F_{\mu\nu} \tilde{F}^{\mu\nu} + q \frac{m^2 c^2}{\hbar^2} (\Phi_\mu \Psi^\mu) (\Psi_\nu \Psi^\nu)^{q-1} - \frac{1}{2} F_{\mu\nu}^* \tilde{F}^{\mu\nu*} + q \frac{m^2 c^2}{\hbar^2} (\Phi_\mu^* \Psi^{\mu*}) (\Psi_\nu^* \Psi^{\nu*})^{q-1} \right\}, \quad (4)$$

where  $A$  is a multiplicative factor that may depend on the total energy and volume (in the case the fields are confined in a finite volume  $V$ ). Moreover, we are adopting the standard index-summation convention [19], and the tensors above are given by

$$F_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu ; \quad \tilde{F}^{\mu\nu} = \partial^\mu \Psi^\nu - \partial^\nu \Psi^\mu . \quad (5)$$

The Euler-Lagrange equations for the vector field  $\Phi_\mu$  [19],

$$\frac{\partial \mathcal{L}}{\partial \Phi_\mu} - \partial_\nu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\nu \Phi_\mu)} \right] = 0 , \quad (6)$$

lead to

$$\nabla^2 \Psi^\mu = \frac{1}{c^2} \frac{\partial^2 \Psi^\mu}{\partial t^2} + q \frac{m^2 c^2}{\hbar^2} \Psi^\mu (\Psi_\nu \Psi^\nu)^{q-1} , \quad (7)$$

where we have used the Lorentz condition  $\partial_\nu \Psi^\nu = 0$  [19, 21]. In a similar way, the Euler-Lagrange equations for the vector field  $\Psi_\mu$

$$\frac{\partial \mathcal{L}}{\partial \Psi_\mu} - \partial_\nu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\nu \Psi_\mu)} \right] = 0 , \quad (8)$$

yield

$$\nabla^2 \Phi_\mu = \frac{1}{c^2} \frac{\partial^2 \Phi_\mu}{\partial t^2} + q \frac{m^2 c^2}{\hbar^2} \left[ \Phi_\mu (\Psi_\nu \Psi^\nu)^{q-1} + 2(q-1) \Psi_\mu (\Phi_\nu \Psi^\nu) (\Psi_\nu \Psi^\nu)^{q-2} \right] . \quad (9)$$

One should notice that the equations above recover the linear Proca equations [18, 19, 21] in the particular limit  $q = 1$ , in which case  $\Phi^\mu(\vec{x}, t) = \Psi^{\mu*}(\vec{x}, t)$ . Furthermore, in the one-

component limit, Eq. (7) recovers the nonlinear Klein-Gordon equation recently proposed in Ref. [1],

$$\nabla^2 \Psi(\vec{x}, t) = \frac{1}{c^2} \frac{\partial^2 \Psi(\vec{x}, t)}{\partial t^2} + q \frac{m^2 c^2}{\hbar^2} [\Psi(\vec{x}, t)]^{2q-1} , \quad (10)$$

whereas Eq. (9) reduces to

$$\nabla^2 \Phi(\vec{x}, t) = \frac{1}{c^2} \frac{\partial^2 \Phi(\vec{x}, t)}{\partial t^2} + q(2q-1) \frac{m^2 c^2}{\hbar^2} \Phi(\vec{x}, t) [\Psi(\vec{x}, t)]^{2(q-1)} , \quad (11)$$

which corresponds to the additional nonlinear Klein-Gordon equation, associated with the field  $\Phi(\vec{x}, t)$ , found by means of a Lagrangian approach in Ref. [7].

For general  $q$ , the fields  $\Psi^\mu(\vec{x}, t)$  and  $\Phi^\mu(\vec{x}, t)$  are distinct, and the solutions of Eqs. (7) and (9) may be written in terms of a  $q$ -plane wave, similarly to the recent nonlinear proposals of quantum equations [1, 2, 7]. In fact, one has that

$$\Psi^\mu(\vec{x}, t) = a^\mu \exp_q \left[ \frac{i}{\hbar} (\vec{p} \cdot \vec{x} - Et) \right] , \quad (12)$$

$$\Phi^\mu(\vec{x}, t) = a^\mu \left\{ \exp_q \left[ \frac{i}{\hbar} (\vec{p} \cdot \vec{x} - Et) \right] \right\}^{-(2q-1)} , \quad (13)$$

satisfy Eqs. (7) and (9), provided that the coefficients are restricted to  $a_\mu a^\mu = 1$ . The solutions above are expressed in terms of the  $q$ -exponential function  $\exp_q(u)$  that emerges in nonextensive statistical mechanics [8], which generalizes the standard exponential, and for a pure imaginary  $iu$ , it is defined as the principal value of

$$\exp_q(iu) = [1 + (1-q)iu]^{1/(1-q)} ; \quad \exp_1(iu) \equiv \exp(iu) , \quad (14)$$

where we used  $\lim_{\epsilon \rightarrow 0} (1 + \epsilon)^{1/\epsilon} = e$ . Moreover, considering these solutions, one obtains the energy-momentum relation

$$E^2 = p^2 c^2 + m^2 c^4 , \quad (15)$$

from both Eqs. (7) and (9), for all  $q$ . Note that, in contrast to what happens with the standard linear Proca equation, the four equations appearing both in (7) and in (9), besides being nonlinear, are coupled. It is remarkable that these sets of four nonlinear coupled partial differential equations admit exact time dependent solutions of the  $q$ -plane wave form that are consistent with the relativistic energy-momentum relation (15).

Now, if one introduces the probability density as

$$\rho(\vec{x}, t) = \frac{1}{2} (\Phi_\mu \Psi^\mu + \Phi_\mu^* \Psi^{\mu*}) , \quad (16)$$

the solutions of Eqs. (12) and (13) yield

$$\rho(\vec{x}, t) = 1 - \frac{(1-q)^2}{\hbar^2} (\vec{p} \cdot \vec{x} - Et)^2 , \quad (17)$$

which require, for positiveness,

$$|(1-q)(\vec{p} \cdot \vec{x} - Et)| < \hbar . \quad (18)$$

### III. MASSLESS PARTICLES: $q$ -PLANE WAVES AS SOLUTIONS OF MAXWELL EQUATIONS

The linear Proca equations are usually considered as appropriate for describing vectorial bosons, or massive photons [19, 21]. Now, if one considers the limit  $m \rightarrow 0$  in the Lagrangian of Eq. (4), one eliminates its nonlinear contributions, recovering the electromagnetic Lagrangian without sources [21]. In this case, both Eqs. (7) and (9) reduce to the standard linear wave equation, described in terms of a single vector field (i.e.,  $\Phi^\mu(\vec{x}, t) = \Psi^{\mu*}(\vec{x}, t)$ ),

$$\nabla^2 \Psi^\mu = \frac{1}{c^2} \frac{\partial^2 \Psi^\mu}{\partial t^2} . \quad (19)$$

As usual, considering  $\vec{p} = \hbar \vec{k}$  and  $E = \hbar \omega$ , one verifies easily that any twice-differentiable function of the type  $f(\vec{k} \cdot \vec{x} - \omega t)$  is a solution of the wave equation. In what follows we will explore the  $q$ -plane wave of Eq. (12) as such a solution; our analysis is based on some

properties of this solution, which are relevant from the physical point of view: (i) It presents an oscillatory behavior; (ii) It is localized for certain values of  $q$ . Indeed, for  $q \neq 1$  the  $q$ -exponential  $\exp_q(iu)$  is characterized by an amplitude  $r_q \neq 1$  [57],

$$\exp_q(\pm iu) = \cos_q(u) \pm i \sin_q(u) , \quad (20)$$

$$\cos_q(u) = r_q(u) \cos \left\{ \frac{1}{q-1} \arctan[(q-1)u] \right\} , \quad (21)$$

$$\sin_q(u) = r_q(u) \sin \left\{ \frac{1}{q-1} \arctan[(q-1)u] \right\} , \quad (22)$$

$$r_q(u) = [1 + (1-q)^2 u^2]^{1/[2(1-q)]} , \quad (23)$$

so that  $r_q(u)$  decreases for increasing arguments, if  $q > 1$ . From Eqs. (20)–(23) one notices that  $\cos_q(u)$  and  $\sin_q(u)$  can not be zero simultaneously, yielding  $\exp_q(\pm iu) \neq 0$ ; moreover,  $\exp_q(iu)$  presents further peculiar properties,

$$[\exp_q(iu)]^* = \exp_q(-iu) = [1 - (1-q)iu]^{1/(1-q)} , \quad (24)$$

$$\exp_q(iu)[\exp_q(iu)]^* = [r_q(u)]^2 = [1 + (1-q)^2 u^2]^{1/(1-q)} , \quad (25)$$

$$\exp_q(iu_1) \exp_q(iu_2) = \exp_q[iu_1 + iu_2 - (1-q)u_1 u_2] , \quad (26)$$

$$\{[\exp_q(iu)]^\alpha\}^* = \{[\exp_q(iu)]^*\}^\alpha = [\exp_q(-iu)]^\alpha , \quad (27)$$

for any  $\alpha$  real. By integrating Eq. (25) from  $-\infty$  to  $+\infty$ , one obtains [58],

$$\mathcal{I}_q = \int_{-\infty}^{\infty} du [r_q(u)]^2 = \frac{\sqrt{\pi} \Gamma\left(\frac{3-q}{2(q-1)}\right)}{(q-1) \Gamma\left(\frac{1}{q-1}\right)} , \quad (28)$$

leading to the physically important property of square integrability for  $1 < q < 3$ ; as some simple typical examples, one has  $\mathcal{I}_{3/2} = \mathcal{I}_2 = \pi$ . One should notice that this integral diverges in both limits  $q \rightarrow 1$  and  $q \rightarrow 3$ , as well as for any  $q < 1$ . Hence, the  $q$ -plane wave of Eq. (12) presents a modulation, typical of a localized wave, for  $1 < q < 3$ .

Then, identifying the components of the vector of Eq. (2) with the scalar and vector potentials,  $\Psi^\mu \equiv (\phi, \vec{A})$ , from which one obtains the fields [21],

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = \vec{\nabla} \times \vec{A}, \quad (29)$$

one verifies that the wave equations for these potentials are equivalent to the Maxwell equations in the absence of sources.

Hence, we now consider the following solutions for each Cartesian component  $j$  ( $j = x, y, z$ ) of the electromagnetic fields,

$$E_j(\vec{x}, t) = E_{0j} \exp_q \left[ i(\vec{k} \cdot \vec{x} - \omega t) \right]; \quad B_j(\vec{x}, t) = B_{0j} \exp_q \left[ i(\vec{k} \cdot \vec{x} - \omega t) \right], \quad (30)$$

which satisfy the wave equation, for each component, provided that  $\omega = c|\vec{k}|$ . Writing the wave vector as  $\vec{k} = k\vec{n}$ , where  $\vec{n}$  represents a unit vector along the wave-propagation direction, the Maxwell equations associated with the divergence of the fields  $\vec{E}$  and  $\vec{B}$  yield respectively,

$$\vec{n} \cdot \vec{E}_0 = 0; \quad \vec{n} \cdot \vec{B}_0 = 0, \quad (31)$$

implying that the fields  $\vec{E}$  and  $\vec{B}$  are both perpendicular to the direction of propagation. In addition to this, considering the solutions of Eq. (30) in Faraday's Law,

$$\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad \Rightarrow \quad \vec{B}_0 = \vec{n} \times \vec{E}_0, \quad (32)$$

whereas doing the same in Ampère's Law,

$$\vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 0 \quad \Rightarrow \quad \vec{E}_0 = -\vec{n} \times \vec{B}_0. \quad (33)$$

The results above show that an electromagnetic wave defined by Eq. (30) corresponds to a transverse wave, similarly to the plane-wave solution.

Now, let us consider the Poynting vector,

$$\vec{S} = \frac{1}{2} \frac{c}{4\pi} \left( \vec{E} \times \vec{B}^* + \vec{E}^* \times \vec{B} \right) = \frac{c}{4\pi} |\vec{E}_0|^2 \left[ 1 + (1-q)^2 (\vec{k} \cdot \vec{x} - \omega t)^2 \right]^{\frac{1}{1-q}} \vec{n}, \quad (34)$$

as well as the energy density,

$$u(\vec{x}, t) = \frac{1}{16\pi} \left( \vec{E} \cdot \vec{E}^* + \vec{B} \cdot \vec{B}^* \right) = \frac{1}{8\pi} |\vec{E}_0|^2 \left[ 1 + (1-q)^2 (\vec{k} \cdot \vec{x} - \omega t)^2 \right]^{\frac{1}{1-q}}. \quad (35)$$

One sees from both expressions above an important difference with respect to those associated with the standard plane-wave solution [21], given by a factor, which is essentially a  $q$ -Gaussian of the argument  $|\vec{k} \cdot \vec{x} - \omega t|$ . As a consequence of this factor, one has that

$$\left( \frac{\partial u}{\partial t} \right)_{\vec{x}} = \frac{1}{4\pi} |\vec{E}_0|^2 \left[ 1 + (1-q)^2 (\vec{k} \cdot \vec{x} - \omega t)^2 \right]^{\frac{q}{1-q}} (q-1) \omega (\vec{k} \cdot \vec{x} - \omega t), \quad (36)$$

which leads to the interesting result,  $(\partial u / \partial t)_{\vec{x}} < 0$ , for a  $q$ -plane wave with  $q > 1$ , if  $\vec{k} \cdot \vec{x} < \omega t$ . This result is directly related with the fact that the amplitude of the wave decreases for increasing arguments, for  $q > 1$ , according to Eqs. (20)–(23). Notice that  $(\partial u / \partial t)_{\vec{x}} = 0$  for  $q = 1$ , as a consequence of the fact that the standard plane wave fills the whole space.

### A. Physical Application: A $q$ -Plane Wave in a Waveguide

Let us now consider the propagation of a  $q$ -plane wave in an infinite rectangular waveguide, adjusted appropriately along the wave-propagation direction, which will be chosen herein to be the  $\vec{x}$ -axis, i.e.,  $\vec{k} \cdot \vec{x} = k_x x$ . The total energy carried by the  $q$ -plane wave can be calculated from Eq. (35),

$$U = \int d\vec{x} u(\vec{x}, t) = \frac{\sigma}{8\pi} |\vec{E}_0|^2 \int_{-\infty}^{\infty} dx \left[ 1 + (1-q)^2 (k_x x - \omega t)^2 \right]^{\frac{1}{1-q}}, \quad (37)$$

where  $\sigma$  represents the area of the transverse section of the waveguide. The integral above may be calculated by means of a change of variables,  $v = k_x x - \omega t$ , in such a way to

use Eq. (28),

$$U = \frac{\sigma}{8\sqrt{\pi}} \frac{|\vec{E}_0|^2}{(q-1)k_x} \frac{\Gamma\left(\frac{3-q}{2(q-1)}\right)}{\Gamma\left(\frac{1}{q-1}\right)}, \quad (38)$$

leading to a finite total energy for  $1 < q < 3$ , diverging in the limit  $q \rightarrow 1$ . As a typical particular case, one has  $U = \sigma|\vec{E}_0|^2/(8k_x)$ , for  $q = 2$ . Hence, due to its localization in time, the total energy that a detector can absorb from the  $q$ -plane is finite, in contrast to what happens with the standard plane wave. This enables the approach of nonlinear excitations which do not deform in time and should be relevant, e.g., in nonlinear optics and plasma physics.

#### IV. CONCLUSIONS

We proposed a generalized Lagrangian that leads to a nonlinear extension of the Proca field equation. We discussed some of the main features of this nonlinear field theory, focusing on the existence of exact time dependent, localized solutions of the  $q$ -plane wave form. These solutions exhibit soliton-like properties, in the sense of propagating with constant velocity and without changing shape. They have a  $q$ -plane wave form, which is a generalization of the standard complex exponential plane wave solutions of the linear Proca equation. The  $q$ -plane waves have properties suggesting that they describe free particles of a finite mass  $m$ : they are compatible with the celebrated Einstein-Planck-de Broglie connection between frequency, wave number, energy, and momentum, satisfying the relativistic relation  $E = p^2c^2 + m^2c^4$ .

In the limit  $q \rightarrow 1$ , the present nonlinear Proca field theory reduces to the Maxwell linear electrodynamics. We see then that the  $q$ -deformation associated with the nonextensive formalism turns Proca's linear field theory into its nonlinear generalization, but leaves Maxwell electrodynamics invariant. As already mentioned, for each value  $q \neq 1$  the associated nonlinear Proca equation admits  $q$ -plane wave solutions. In the  $q \rightarrow 1$  limit, however, the  $q$ -plane wave solutions are solutions of Maxwell equations for all  $q$ .

The nonlinear Proca equation here introduced, together with the nonlinear Dirac and Klein-Gordon equations previously advanced in [1, 7], provide the main ingredients of a

nonlinear generalization of the basic relativistic field equations for particle dynamics inspired in the  $q$ -thermostatistical formalism. These equations share a family of exact time dependent solutions: the  $q$ -plane waves. The present discussion suggests several possible directions of future research, such as to study more complex wave-packet solutions, and to consider interactions. Some progress in these directions has been achieved, in a non-relativistic setting, in the case of a nonlinear Schroedinger equation with a power-law nonlinearity in the kinetic energy term [3–5]. It would be interesting to extend these results to relativistic scenarios. Another relevant issue for future exploration concerns gauge invariance. As happens with the standard linear Proca equation, the present nonlinear Proca equation is not gauge invariant, due to the presence of the mass term. It would be interesting to explore whether this symmetry can be restored by adding more dynamics to the nonlinear Proca field theory, as is done in the linear theory via the Stueckelberg procedure [23]. Any new developments along these lines will be very welcome.

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