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Transient superdiffusion in random walks with a q-exponentially decaying memory profile

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HIGHLIGHTS

- The Fokker–Planck equation is derived with an implicit q-dependence associated with the memory size.
- These results broaden our knowledge on the importance of the diffusive properties of the walker.
- The results shown here pave the way for treating other non-Markovian memory patterns in future work.

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ABSTRACT

We propose a random walk model with q-exponentially decaying memory profile. The q-exponential function is a generalization of the ordinary exponential function. In the limit $q \to 1$, the q-exponential becomes the ordinary exponential function. This model presents a Markovian diffusive regime that is characterized by finite memory correlations. It is well known, that central limit theorems prohibit superdiffusion for Markovian walks with finite variance of step sizes. In this problem we report the outcome of a transient superdiffusion for finite sized walks.

1. Introduction

The random walk model is widely used to describe microscopic transport processes which are characterized by intrinsic randomness. This model and its generalization, the Continuous Time Random Walk (CTRW) set out by E.W. Montroll and G.H. Weiss, are important tools for the study of physical, chemical and economical phenomena [1–3]. One important problem of the random walks is the inclusion of memory correlations in dynamics. The corresponding non-Markovian processes represent a challenge for physicists and mathematicians. The memory effects are incorporated heuristically for physical observables [4] and important for some phenomena, such as turbulence [5] and anomalous diffusion [6]. In organic materials the CTRW is used to describe anomalous transport phenomena [7,8]. In industrial applications, random walk models are important to describe and build new devices, such as amorphous materials used in xerography machines [9] and optical-memory devices [10]. The type of diffusion of a random walker can be determined by studying the asymptotic scaling of the

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mean square displacement (MSD) with time. The MSD is defined by $\langle \Delta x^2 \rangle = \langle (x - \langle x \rangle)^2 \rangle \sim t^{2H}$ where the asymptotic scaling exponent is known as the Hurst exponent [1–3]. The regime is anomalous for $H \neq 1/2$ and normal for $H = 1/2$. For $H < 1/2$ ($H > 1/2$) the regime is termed subdiffusive (superdiffusive). In the model described in the next section we use the Hurst exponent to determine the diffusion regimes of the random walker. We briefly discuss our results in relation to q-Central Limit Theorem (q-CLT) [11,12].

2. The model

We study a model derived from the elephant random walk model (ERW) proposed by G.M. Schütz and S. Trimper [4]. The ERW is termed non-Markovian because the walker keeps a record of the entire history of the walk. The ERW model presents a transition from a normally diffusive scape regime to a superdiffusive regime at $p_c = 3/4$. Some variants of this model have been proposed, such as the (truncated) 'Gaussian random walk' [13], the random walk with exponential memory profile [14] (with exact and numerical solutions leading to an unexpected superdiffusive regime) and the Alzheimer random walk model which exhibits amnestically induced superdiffusion and log-periodic corrections to scaling [15]. The random walk driven by a q-exponential memory is inspired by the recently proposed exponential memory profile model [14].

We use the same notation introduced in the original ERW [4]. The elephant random walk is described as follows: the walker starts at the origin x_0 at time $t_0 = 0$, and at $t > 0$ it moves one step to the left or to the right. The walker keeps a record of the entire history of the walk and the process is described by the stochastic equation

$$x_{t+1} = x_t + \sigma_{t+1}. \tag{1}$$

An equiprobable time t' is chosen from the previous time set $\{1, 2, \dots, t\}$ at a time $t + 1$. The stochastic variable $\sigma_{t+1} = \pm 1$ is then chosen by the following rule

$$\sigma_{t+1} = \begin{cases} +\sigma_{t'} & \text{with probability} \quad p \\ -\sigma_{t'} & \text{with probability} \quad 1-p. \end{cases} \tag{2}$$

The first step is always chosen to the right, i.e., $\sigma_1 = +1$. The position at time t thus follows

$$x_t = \sum_{t'=1}^{t} \sigma_{t'} \tag{3}$$

and the second moment is given by Ref. [14]

$$\langle x^2(t) \rangle = \begin{cases} \dfrac{t}{3-4p}, & p < \dfrac{3}{4} \\ t \ln t, & p = \dfrac{3}{4} \\ \dfrac{t^{4p-2}}{(3-4p)\Gamma(4p-2)}, & p > \dfrac{3}{4} \end{cases} \tag{4}$$

which are exact relations valid in the asymptotic limit. The diffusion regime is therefore normal for $p < 3/4$ and superdiffusive for $p > 3/4$. For $p = 3/4$ the walk is marginally superdiffusive. The exact propagator is reported to be a Gaussian distribution for all regimes [4], and is described by the following equation

$$P(x, t) = \frac{1}{\sqrt{4\pi D(t)}} \exp\left(-\frac{(x - \langle x(t) \rangle)^2}{4\pi D(t)} \right) \tag{5}$$

with

$$D(t) = \frac{1}{8p-6} \left[\left(\frac{t}{t_0} \right)^{4p-3} - 1 \right] \tag{6}$$

where $D(t)$ is the diffusion coefficient. It was found recently that the superdiffusive regime is actually characterized by a non-Gaussian distribution [16].

Our model is proposed as a random walk capable to remember past memory events with a memory profile described by the q-exponential probability distribution function. In the ERW model the previous time t' is chosen from a uniform distribution, whereas in this model, a number from the set $\{1, 2, \dots, t\}$ is randomly chosen from a q-exponential distribution, written as

$$P(t, t') = A \exp_q(-\lambda(t - t')) \tag{7}$$

where

$$\exp_q(-\lambda(t - t')) = [1 + (1 - q)(-\lambda(t - t'))]^{\frac{1}{1-q}}, \tag{8}$$

with decay constant $\lambda = 1$. The parameter A is normalization constant and q is a dimensionless parameter which adjusts the shape of the function.

In the next section we discuss the results obtained by mapping the q-exponential model onto the rectangular memory model with recent memory. We show that the q-exponential memory profile model leads to a transient superdiffusion for suitable values of q.

3. Results

We estimate suitable q values for specific input times by mapping the q-exponential model to the rectangular memory profile model through the effective memory length equation [14],

$$L = \int_0^t \frac{P_\lambda(t', t)}{P_{\max}(t', t)} \mathrm{d}t' \tag{9}$$

resulting in the following relations

$$L = \begin{cases} \dfrac{1}{\lambda(2-q)}, & 0 \le q < 2 \\ \dfrac{\ln t}{\lambda}, & q = 2 \\ Ct^{\frac{q-2}{q-1}}, & q > 2. \end{cases} \tag{10}$$

These relations are valid in the asymptotic limit and the constant C is defined as $C = [\lambda(q-1)]^{\frac{q-2}{q-1}}/\lambda(q-2)$. Note that for the case of $0 \leqslant q < 1$, the q-exponential (Eq. (8)) is defined only in the interval $t - 1/[\lambda(1-q)] \le t' \le t$.

It is well known that exponentially decaying memory correlations cannot lead to superdiffusion. However, superdiffusion can occur if a time dependent decay constant is associated with the exponential behavior (see Ref. [14]). As $q \to 1$ the q-exponential becomes the ordinary exponential function, i.e.,

$$\lim_{q \longrightarrow 1} \exp_q(-(t-t')) = \exp(-(t-t')). \tag{11}$$

The Central Limit Theorem (CLT) is of fundamental importance in Statistical Mechanics and a cornerstone in Statistics. According to q-CLT in the limit $q \to 1$ the random walk is completely Markovian and thus no superdiffusion should arise [11,12].

We can obtain the Fokker–Planck (FP) equation for the square model equivalent to the q-exponential decaying model by following the exact same lines of Ref. [14]. This involves the definition of a discrete conditional probability that the walker is at the position x at time $t+1$ given the earlier position x_0 at $t = 0$ (see Ref. [14] for details). Then, for large $L = ft$ (where $0 \le f \le 1$ is a fraction of the recent memory) we can easily show that an approximate FP for the propagator can be written as

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 P(x,t)}{\partial x^2} - \frac{\alpha}{ft}\frac{\partial}{\partial x}\{xP(x,t) - x[P(x,[1-f]t) + g_{1-f}]\}, \tag{12}$$

with the definition of the stochastic function $g_{1-f} \equiv g_{1-f}[x, (1-f)t]$ satisfying $\int_{-\infty}^{+\infty} x g_{1-f} \mathrm{d}x = 0$. Note that the dependence on q is implicit in f according to Eq. (10).

Fig. 1 shows the behavior of the Hurst exponent with $1/q$ for different values of the time step length t_s, namely, 10^3, 10^5 and 10^7. We see that $H \to 1/2$ as $t \to \infty$. The different values for the time step length show the transient superdiffusion effect more clearly. The mathematical relations in (Eq. (10)) are then used to find a transient superdiffusion time related with a specific value of q. It is known that, for the memory profile consisting of a uniform squared memory pattern of memory length $L = ft$, there is no superdiffusion for $f < 1/2$ [17]. So we look for a suitable time for the onset of a transient superdiffusion. For $q > 2$, we found $L = Ct^{\frac{(q-2)}{(q-1)}}$, and if we get $L = t/2$, we have $Ct^{\frac{(q-2)}{(q-1)}} = t/2$, and therefore we can write the transient superdiffusion time as, $t = (2C)^{q-1}$, which is used to find the relation between q and t. For example, for an input transient time of the order of $t \sim 10^6$, we find $q \sim 25$. This means that for $q \sim 25$, $t \sim 10^6$ is the time at which superdiffusion will start to change regime, from a superdiffusive ($H > 1/2$) to a diffusive regime ($H = 1/2$). Thus, in the asymptotic limit all transient superdiffusion vanish and the walk becomes Markovian.

Figs. 2 and 3 show the behavior of the Hurst exponent for 10^7 time steps and 10^4 runs and for a wide range of q values. We see that the transient superdiffusion disappears for small values of q (typically $q \lesssim 3/2$). For large values of p and q, the Hurst exponent is greater than 1/2. For $q = 3/2$, where transient superdiffusion is first observed, the transient regime starts at $p \sim 0.95$, and for the highest value $q = 25$ the transient transition starts at $p \sim 0.75$. It is interesting to note that there are two possible case limits in this model: one in which t is kept fixed and $q \to \infty$ and another in which q is fixed and $t \to \infty$. In the first case we get the ERW model in the asymptotic limit. In the second case $H \to 1/2$ even for large values of t.

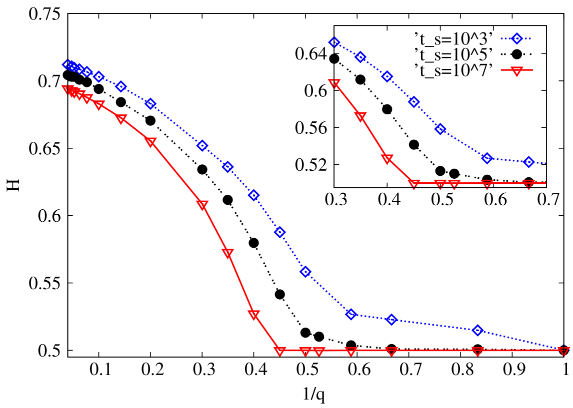

Fig. 1. Plot of the Hurst exponent H for $p = 0.85$ and several values of t_s. The averages are calculated for 10^3 (squares), 10^5 (circles) and 10^7 (triangles) time steps with 10^4 runs. The values for the Hurst exponents show the transient superdiffusion phenomena. From the inset, it becomes evident that the Hurst exponent is larger for small times.

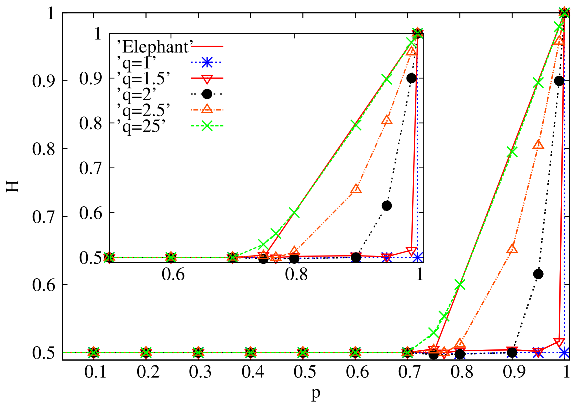

Fig. 2. Plot of the Hurst exponent H as a function of p. The averages are calculated for 10^7 time steps with 10^4 runs. Several values of q are shown. The Elephant Model is shown for comparison (continuous red line). For $q = 1$ (asterisks) the walk is the usual Markovian walk ($H = 1/2$). The transient superdiffusion phenomena ($H > 1/2$) first appear for $q = 3/2$ (inverted triangles), and grow for $q = 2$ (circles), $q = 5/2$ (triangles) and $q = 25$ (green crosses).

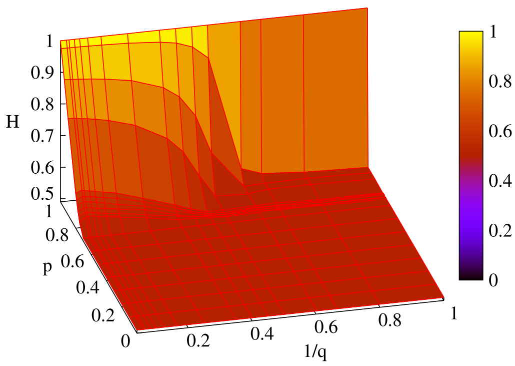

Fig. 3. Plot of the Hurst exponent H as a function of both parameters q and p for the q-exponential memory model. The averages are calculated for 10^7 time steps with 10^4 runs and $1 \leqslant q \leqslant 25$, according with Eq. (10). As the parameter q grows the transient superdiffusion becomes more evident.

4. Conclusions

We numerically study a random walk model with a q-exponentially decaying memory profile along with the second moment $\langle x^2 \rangle \sim t^{2H}$, to estimate the Hurst exponent. Although no superdiffusion should be expected according to the q-Central Limit Theorem, a transient is surprisingly found for $q \gtrsim 3/2$. The transient superdiffusion disappears in accordance with the q-CLT. The Fokker–Planck equation is also derived with an implicit q-dependence associated with the memory size. We only consider the case $\lambda = 1$. It is easy to notice that $\lambda < 1$ only increases the memory range and therefore favors superdiffusion, leading to higher values for the corresponding Hurst exponents. The opposite happens for $\lambda > 1$. However, no qualitative changes should occur for $\lambda \neq 1$. Exact results associated with non-Markovian regimes are rare. It is thus important to be able to map non-linear memory search algorithms to analytically treatable ones. These results broaden our knowledge on the importance of the memory search mechanism and its overall effects on the diffusive properties of the walker. The results shown here pave the way for treating other non-Markovian memory patterns in future work.

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