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Nonlinear Kramers equation associated with nonextensive statistical mechanics

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Stationary and time-dependent solutions of a nonlinear Kramers equation, as well as its associated nonlinear Fokker-Planck equations, are investigated within the context of Tsallis nonextensive statistical mechanics. Since no general analytical time-dependent solutions are found for such a nonlinear Kramers equation, an ansatz is considered and the corresponding asymptotic behavior is studied and compared with those known for the standard linear Kramers equation. The H-theorem is analyzed for this equation and its connection with Tsallis entropy is investigated. An application is discussed, namely the motion of *Hydra* cells in two-dimensional cellular aggregates, for which previous measurements have verified q-Gaussian distributions for velocity components and superdiffusion. The present analysis is in quantitative agreement with these experimental results.

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I. INTRODUCTION

Recent attempts for an appropriate description of several complex systems demanded the introduction of a wide variety of nonlinear (NL) equations [1,2]. As an immediate consequence, difficulties appeared in finding exact solutions, so approximative analytical methods and numerical procedures have been used. In these later cases, many researches benefitted from advances in computer technology, leading to considerable progresses in several areas of physics, so problems that have remained intractable for decades are now being dealt numerically. Among many, one could mention some areas where these types of equations occur frequently, like nonlinear optics, superconductivity, plasma physics, and nonequilibrium statistical mechanics.

Very frequently, NL equations are proposed as generalizations of linear ones, so these are recovered as particular cases. Among many possible ways, such generalizations were carried in the literature mostly in two different procedures: (i) the addition of new NL terms to a linear equation and (ii) the modification of exponents of existing linear terms. This second procedure has been much used within the theory of nonextensive statistical mechanics [3–6], which emerged from the generalization of the Boltzmann-Gibbs entropy, by introducing a real index q, such as to recover the former in the limit $q \to 1$ [3]. Similarly, a power $(2-q)$ in the probability of the diffusion term leads to a NL Fokker-Planck equation (NLFPE) [7,8] capable of explaining interesting physical phenomena related to anomalous diffusion [9] (for which $q \neq 1$). Additionally, linear inhomogeneous Fokker-Planck equations were proposed as well, and some were associated to nonextensive statistical mechanics [10]. Since then, a wide variety of NLFPEs were investigated in the literature [11], motivated by a description of several physical phenomena related to anomalous diffusion in multiple dimensions and anisotropic media [12–16].

The H-theorem represents one of the most important results of nonequilibrium statistical mechanics, guaranteeing that a system will approach an equilibrium state after a long-time evolution. One possible proof of this theorem can be carried by using a Fokker-Planck equation and introducing an associated entropic form, appropriated for the physical system under investigation. According to this, the linear Fokker-Planck equation is usually related to the Boltzmann-Gibbs entropy either by means of the H-theorem or by comparing its solution with the distribution that comes from an entropy maximization procedure [17–19]. In a similar manner, generalized forms of the H-theorem have been worked out recently in such a way to relate NLFPEs with other entropic forms [11,20–37].

The Kramers equation may be considered a type of Fokker-Planck equation describing the time evolution of a joint probability distribution $P(\vec{x},\vec{v},t)$ for both positions ($\{\vec{x}_i\}$) and velocities ($\{\vec{v}_i\}$) of a system of particles (see, e.g., Refs. [19,38] for a discussion of the linear Kramers equation). In the linear case, by integrating over velocities one obtains as a first approximation the standard Fokker-Planck equation, where the distribution $P(\vec{x},t)$ appears as a marginal probability. Although nonlinear forms of the Kramers equation have been proposed in the literature [11,29,39], they have not yet been explored in detail; essentially, only their stationary-state solutions were studied.

In this work we investigate a nonlinear Kramers equation (NLKE) associated with nonextensive statistical mechanics. Apart from the stationary state, time-dependent solutions are also analyzed. For completeness, in the next section we present a short review of the linear Kramers equation, showing how it is related with Langevin equations, and discuss its solutions. In Sec. III we introduce the NLKE to be studied, as well as its associated Langevin equations. The stationary-state solutions are found, an ansatz for the time-dependent solutions is presented, and a discussion about H-theorems is performed as well. In Sec. IV we discuss an application, namely a biological system, composed by single *Hydra* cells, which move in a two-dimensional space [40,41]. The velocity distributions of the centers of mass of these cells have been measured

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previously and shown to follow q-Gaussian distributions [41], thus representing a suitable system for being analyzed within the present approach; moreover, anomalous diffusion was also identified in Ref. [40]. We show that the reduced velocity distribution and the anomalous diffusion can be related to a time-dependent solution of the present NLKE. We verified a quantitative agreement between this approach and the experimental estimates. Finally, in Sec. V we present our conclusions.

II. LINEAR KRAMERS EQUATION

Let us consider a given system described by a set of n variables, $\vec{x}(t) \equiv [x_1(t),x_2(t),\ldots,x_n(t)]$, following Langevin equations,

$$\frac{dx_i}{dt} = f_i + \eta_i, \qquad \langle \eta_i(t)\rangle = 0, \quad \langle \eta_i(t)\eta_i(t')\rangle = \Gamma_{ij}\delta(t-t'), \tag{2.1}$$

where $\{f_i(\vec{x},t)\}$ are general functions and $\{\eta_i(t)\}$ $(i = 1,2,\cdots,n)$ are stochastic variables, corresponding to an additive white noise. From Eqs. (2.1) one may obtain a Fokker-Planck equation for a probability density $P(\vec{x},t)$ at time t [18,19,38],

$$\frac{\partial P(\vec{x},t)}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{n}\Gamma_{ij}\frac{\partial^2 P(\vec{x},t)}{\partial x_i \partial x_j} - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}[f_i P(\vec{x},t)], \tag{2.2}$$

where the matrix elements $\{\Gamma_{ij}\}$ correspond to diffusion coefficients, covering the most general anisotropic situation.

Herein we focus attention on a particle with mass m, in a one-dimensional space, subjected to an external force F_x, a drag force $-\alpha v$ $(\alpha > 0)$, as well as a random force $\eta(t)$. The associated set of Langevin equations is given by

$$m\frac{dv}{dt} = -\alpha v + F_x + \eta, \qquad \frac{dx}{dt} = v, \tag{2.3}$$

where $\langle \eta(t)\rangle = 0$ and $\langle \eta(t)\eta(t')\rangle = \Gamma\delta(t-t')$. Comparing Eqs. (2.1) and (2.3) one sees that in the above case $n = 2$, corresponding to the set of variables $\vec{x}(t) \equiv (x,v)$, where the random force in the second equation is null; moreover, we can identify $\Gamma_{xx} = \Gamma_{xv} = \Gamma_{vx} = 0$ and $\Gamma_{vv} = \Gamma/m^2$. The associated Fokker-Planck equation for $P(x,v,t)$, usually referred to as Kramers equation, becomes

$$\frac{\partial P(x,v,t)}{\partial t} = \frac{\Gamma}{2m^2}\frac{\partial^2 P(x,v,t)}{\partial v^2} - \frac{\partial}{\partial v}\left[\left(-\frac{\alpha v}{m} + \frac{F_x}{m}\right)P(x,v,t)\right] - \frac{\partial}{\partial x}[vP(x,v,t)]. \tag{2.4}$$

One should notice that Eq. (2.4), for the joint probability distribution $P(x,v,t)$, differs from a standard two-dimensional Fokker-Planck equation, since the x and v variables are not symmetric.

Below we discuss two particular cases that arise from the Langevin equations in Eq. (2.3). First, when $m dv/dt \ll \alpha v$, one may neglect the contribution $m dv/dt$, such as to obtain the overdamped regime,

$$\frac{dx}{dt} = \frac{F_x}{\alpha} + \frac{\eta}{\alpha}, \tag{2.5}$$

that leads to

$$\frac{\partial P(x,t)}{\partial t} = \frac{\Gamma}{2\alpha^2}\frac{\partial^2 P(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left[\frac{F_x}{\alpha}P(x,t)\right]. \tag{2.6}$$

Identifying $D = \Gamma/(2\alpha^2)$ and $f = F_x/\alpha$, one recovers the usual linear Fokker-Planck equation for $P(x,t)$ [18,19,38]. Another particular case occurs when $F_x = 0$, for which

$$\frac{dv}{dt} = -\frac{\alpha}{m}v + \frac{\eta}{m}. \tag{2.7}$$

A procedure similar to the one used to obtain Eq. (2.6) yields in the present case

$$\frac{\partial P(v,t)}{\partial t} = \frac{\Gamma}{2m^2}\frac{\partial^2 P(v,t)}{\partial v^2} - \frac{\partial}{\partial v}\left[-\frac{\alpha v}{m}P(v,t)\right], \tag{2.8}$$

which corresponds to a reduced Fokker-Planck for the marginal probability distribution $P(v,t)$, associated with the velocities. Indeed, since one has no position dependence in the velocity equation, it is possible to integrate Eq. (2.4) over x to obtain Eq. (2.8). On the other hand, an integration of Eq. (2.4) over v after an approximation (i.e., considering an overdamped regime) leads to Eq. (2.6). In what follows, we review briefly the solutions of the Kramers equation in Eq. (2.4).

A. Stationary solution

For a confining external force independent of time, we obtain a stationary solution for a sufficiently long time. Thus, by imposing $\partial P/\partial t = 0$ in Eq. (2.4), one verifies that

$$P(x,v) = A\exp\left\{-\frac{2\alpha}{\Gamma}\left[\frac{mv^2}{2} + V(x)\right]\right\}, \tag{2.9}$$

where $V(x) = -\int_0^x F(x')dx'$ and A is a normalization factor. Analogously, for the two particular cases above [Eqs. (2.6) and (2.8)], we have, respectively,

$$P(x) = A'\exp\left[-\frac{2\alpha}{\Gamma}V(x)\right] \tag{2.10}$$

and

$$P(v) = A''\exp\left(-\frac{\alpha m}{\Gamma}v^2\right), \tag{2.11}$$

where, like in Eq. (2.9), A', and A'' are normalization factors. One should mention that the stationary distributions of Eq. (2.10) and Eq. (2.11) may be found also by integrating Eq. (2.9) over v, or x, respectively, leading to the above reduced distributions.

B. Time-dependent solution

For completeness, we write below the time-dependent solution of Eq. (2.4) as usually found in standard books (see, e.g., Refs. [19,38]), expressed in terms of the following Gaussian ansatz:

$$P(\vec{x},t) = \frac{1}{Z(t)}\exp\{-[\vec{x}-\vec{x}_{0i}(t)]G^{-1}(t)[\vec{x}-\vec{x}_{0i}(t)]\}, \tag{2.12}$$

where $Z(t)$, $\vec{x}_{0i}(t)$, and $G^{-1}(t)$ are determined by imposing $P(\vec{x},t)$ to be a solution. In the present case, $G(t)$ corresponds to a 2×2 Green-function matrix, which depends on the coefficients of Eq. (2.4). It should be mentioned that the ansatz above is useful when F_x is linear on one of the variables, x or v (or in both variables). As expected, the above time-dependent solution recovers the stationary solution of Eq. (2.9) in the long-time limit.

In the following sections we will investigate a NLKE associated with nonextensive statistical mechanics, for which the stationary-state and time-dependent solutions discussed above should be recovered as particular limits.

III. NONLINEAR KRAMERS EQUATION

Within the scenario of the nonextensive statistical mechanics, a NLFPE was proposed, being capable of explaining many interesting physical phenomena related to anomalous diffusion [7,8]. Essentially, it consists in a modification of Eq. (2.2) by introducing an exponent in the diffusion term,

$$\frac{\partial P(\vec{x},t)}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{n} \Gamma_{ij} \frac{\partial^2 P^{\nu}(\vec{x},t)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[f_i P(\vec{x},t)], \tag{3.1}$$

where ν is a real positive parameter, directly related to the nonextensive entropic index through $q = 2 - \nu$, i.e., $q < 2$. The linear case is recovered in the limit $\nu \to 1$, and for $\nu \neq 1$ one has a NLFPE. Along the procedure used for obtaining Eq. (3.1), a generalization of Eq. (2.4) was proposed in Refs. [11,39],

$$\begin{aligned}\frac{\partial P(x,v,t)}{\partial t} =& \frac{\Gamma}{2m^2}\frac{\partial^2 P^{\nu}(x,v,t)}{\partial v^2} \\ &- \frac{\partial}{\partial v}\left[\left(\frac{F_v}{m} + \frac{F_x}{m}\right)P(x,v,t)\right] \\ &- \frac{\partial}{\partial x}[vP(x,v,t)],\end{aligned} \tag{3.2}$$

which represents the type of NLKE to be studied herein. As far as we know, the equation above has not been much explored in the literature, and only its stationary solution has been obtained so far [11,39]. Equation (3.2) can be derived from a set of Langevin equations, essentially by following a scheme similar to one used to derive a NLFPE [42]; in this way, one introduces the generalized Langevin equations,

$$\begin{aligned} m\frac{dv}{dt} &= F_v + F_x + P^{(\nu-1)/2}\eta, \\ \frac{dx}{dt} &= v, \end{aligned} \tag{3.3}$$

where $F_v = -dK(v)/dv$, $F_x = -dV(x)/dx$, $\langle \eta(t) \rangle = 0$, and $\langle \eta(t)\eta(t') \rangle = \Gamma\delta(t-t')$. Considering an approach similar to those used to obtain the linear Kramers equation [Eq. (2.4)], one gets also Eq. (3.2).

We emphasize that the potentials $V(x)$ and $K(v)$ may not be necessarily quadratic, although in order to obtain a stationary solution in the long-time limit, they should be confining potentials; hence one may consider more general situations, e.g., $V(x) = \gamma|x|^{b_1}/b_1$ and $K(v) = \alpha|v|^{b_2}/b_2$ (γ, b_1, α, and b_2 all positive) [35]. However, the quadratic potentials are the most commonly used in the literature, for which one can calculate more easily analytical solutions in some cases; therefore, whenever necessary to consider an explicit form for these potentials, we will consider for simplicity the quadratic ones, $K(v) = \alpha v^2/2$ (i.e., $F_v = -\alpha v$) and $V(x) = \gamma x^2/2$ (i.e., $F_x = -\gamma x$).

Like in the linear case, we can also analyze some particular limits. First, when $m dv/dt \ll \alpha v$, one has

$$\frac{dx}{dt} = \frac{F_x}{\alpha} + P^{(\nu'-1)/2}\frac{\eta}{\alpha}, \tag{3.4}$$

which yields

$$\frac{\partial P(x,t)}{\partial t} = \frac{\Gamma}{2\alpha^2}\frac{\partial^2 P^{\nu'}(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left[\frac{F_x}{\alpha}P(x,t)\right], \tag{3.5}$$

which corresponds essentially to there NLFPE for $P(x,t)$ proposed in Ref. [7]. Another interesting particular case occurs in the limit $F_x \to 0$, i.e., the free-particle limit,

$$\frac{dv}{dt} = \frac{F_v}{m} + P^{(\nu''-1)/2}\frac{\eta}{m}, \tag{3.6}$$

leading to a NLFPE for the velocities,

$$\frac{\partial P(v,t)}{\partial t} = \frac{\Gamma}{2m^2}\frac{\partial^2 P^{\nu''}(v,t)}{\partial v^2} - \frac{\partial}{\partial v}\left[\frac{F_v}{m}P(v,t)\right]. \tag{3.7}$$

It is important to note that the exponents ν, ν', and ν'', appearing, respectively, in Eqs. (3.2), (3.5), and (3.7), are not necessarily equal, since they may describe different regimes of the anomalous diffusion. In what follows, these aspects will be discussed in further detail.

A. Stationary-state solutions

In this subsection we analyze stationary-state solutions (for which $\partial P/\partial t = 0$) of Eqs. (3.2), (3.5), and (3.7), introduced above. In all these equations there is a diffusion-like contribution, as well as drift contributions; herein we assume that these later terms confine the particle to a finite region, such that the stationary state is reached after a sufficiently long time.

In this case, the stationary state of Eq. (3.5) is given by

$$\frac{dP^{\nu'}(x)}{dx} = \frac{2\alpha}{\Gamma}F_x P(x), \tag{3.8}$$

whose solution is

$$P(x) = \tilde{A}' \exp_{q'}[-\sigma' V(x)], \tag{3.9}$$

where $q' = 2 - \nu'$, $\sigma' = [2\alpha(\tilde{A}')^{1-\nu'}]/(\nu'\Gamma)$, and $\tilde{A}'$ is a normalization constant. Above, we used the q-exponential definition [5],

$$\exp_q(w) \equiv \begin{cases} [1 + (1-q)w]^{1/(1-q)}, & \text{if} \quad (q-1)w \leqslant 1 \\ 0, & \text{if} \quad (q-1)w > 1 \end{cases}. \tag{3.10}$$

The function above may be written also as $\exp_q(w) = [1 + (1-q)w]_+^{1/(1-q)}$, where $[y]_+ = y$ if $y > 0$, zero otherwise.

In a similar way (i.e., in the stationary regime), Eq. (3.7) leads to

$$\frac{dP^{\nu''}(v)}{dv} = \frac{2m}{\Gamma} F_v P(v), \tag{3.11}$$

and, therefore,

$$P(v) = \tilde{A}'' \exp_{q''}[-\sigma'' K(v)], \tag{3.12}$$

where $q'' = 2 - \nu''$, $\sigma'' = [2m(\tilde{A}'')^{1-\nu''}]/(\nu''\Gamma)$, and $\tilde{A}''$ denotes a normalization constant. In the above solutions the potentials $V(x)$ and $K(v)$ represent conveniently chosen potentials, such as to yield a stationary-state state in the long-time limit.

Now, for the NLKE of Eq. (3.2), we have

$$\frac{\Gamma}{2m^2}\frac{\partial^2 P^{\nu}(x,v)}{\partial v^2} - \frac{\partial}{\partial v}\left[\left(\frac{F_v}{m} + \frac{F_x}{m}\right)P(x,v)\right] - \frac{\partial}{\partial x}[vP(x,v)] = 0, \tag{3.13}$$

which yields

$$P(x,v) = \tilde{A}\exp_q\left\{-\sigma\left[\frac{K(v)}{\alpha} + \frac{V(x)}{m}\right]\right\}, \tag{3.14}$$

with $q = 2 - \nu$, $\sigma = [2\alpha m(\tilde{A})^{1-\nu}]/(\nu\Gamma)$, and $\tilde{A}$ being a normalization constant. Comparing this solution above with Eq. (2.9), one notices that the above stationary distribution presents a similar form, although expressed in terms of a q exponential of both potentials. The distribution above is represented in Fig. 1 in terms of conveniently rescaled dimensionless variables for both $V(x)$ and $K(v)$ harmonic potentials and typical values of q. As expected, these distributions are characterized by a compact support for $q < 1$ and infinite support for $q > 1$.

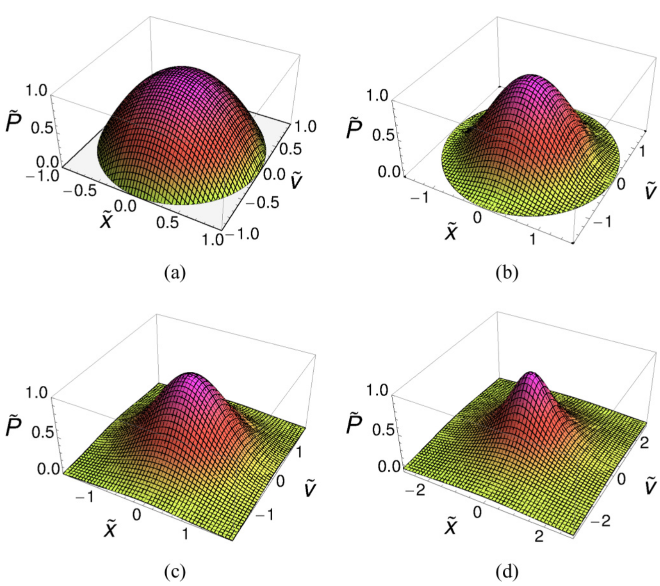

FIG. 1. (Color online) The stationary probability distribution of Eq. (3.14) is exhibited for typical values of q: (a) $q = 0$, (b) $q = 0.6$, (c) $q = 1.1$, and (d) $q = 1.9$. The potentials used were the harmonic ones, $V(x) = \gamma x^2/2$ and $K(v) = \alpha v^2/2$, whereas $\tilde{P}$, $\tilde{x}$, and $\tilde{v}$ represent conveniently rescaled dimensionless variables.

One expects the stationary distribution of Eq. (3.14) to yield Eqs. (3.9) and (3.12) as reduced distributions. Now, in order to perform the integrals analytically, we will consider harmonic potentials in each case. Considering $K(v) = \alpha v^2/2$, one may integrate Eq. (3.14) over v to obtain

$$P(x) = \int_{-\infty}^{\infty} P(x,v)dv = \frac{[1-(1-q)\sigma V(x)]_+^{[1/(1-q)]+1/2}}{\int_{-\infty}^{\infty}[1-(1-q)\sigma V(x)]_+^{[1/(1-q)]+1/2}dx}. \tag{3.15}$$

For this reduced distribution to recover exactly that one presented in Eq. (3.9), namely $P(x) = \tilde{A}'[1-(1-q')\sigma' V(x)]_+^{1/(1-q')}$, it is necessary that

$$q' = \frac{1+q}{3-q} \quad \text{and} \quad \sigma' = \left(\frac{3-q}{2}\right)\sigma. \tag{3.16}$$

This result indicates that q and σ change when one carries an integration over one of the variables; moreover, one notices that only for $q = 1$ is that such changes do not occur.

In a similar way, considering $V(x) = \gamma x^2/2$, one may integrate Eq. (3.14) over x to obtain

$$P(v) = \int_{-\infty}^{\infty} P(x,v)dx = \frac{[1-(1-q)\sigma K(v)]_+^{[1/(1-q)]+1/2}}{\int_{-\infty}^{\infty}[1-(1-q)\sigma K(v)]_+^{[1/(1-q)]+1/2}dv}, \tag{3.17}$$

where, again, for this reduced distribution to represent exactly the one presented in Eq. (3.12), namely $P(v) = \tilde{A}''[1-(1-q'')\sigma'' K(v)]_+^{1/(1-q'')}$, one needs to impose relations involving (q'',σ'') and (q,σ) that are similar to those of Eq. (3.16).

We conclude that, for parabolic potentials in both variables x and v, one has similar reduced distributions, characterized by the same changes in q and σ. This result will be referred to in Sec. IV, where we discuss the motion of *Hydra* cells in a two-dimensional cell aggregate; the theoretical estimate of the anomalous-diffusion exponent will be compared with the corresponding experimental value.

B. Time-dependent solution

Now let us address the time-dependent case; similarly to the proposal of Eq. (2.12), it is natural to consider

$$P(\vec{x},t) = \frac{1}{Z(t)}\exp_q\{-[\vec{x} - \vec{x}_{0i}(t)]G^{-1}(t)[\vec{x} - \vec{x}_{0i}(t)]\}, \tag{3.18}$$

as an ansatz for the nonlinear Kramers equation, where $Z(t)$, $\vec{x}_{0i}(t)$, and $G^{-1}(t)$ are determined by imposing the above $P(\vec{x},t)$ to be a solution of Eq. (3.2).

As a check, we apply the ansatz of Eq. (3.18) for the time-dependent particular cases in Eqs. (3.5) and (3.7), which are essentially equivalent equations if one considers the identifications $x \leftrightarrow v$. Doing this for Eq. (3.7) with $F_v = -\alpha v$, one obtains

$$v_0(t) = v_0(0)\exp\left(-\frac{\alpha t}{m}\right), \tag{3.19}$$

$$Z(t)G(t)^{-1/2} = \text{const.}, \tag{3.20}$$

$$G(t) = G(0)\Bigg(\exp\left[-(1+\nu'')\frac{\alpha t}{m}\right] - \frac{\Gamma\nu'' Z(0)^{1-\nu''}}{m\alpha G(0)} \times\left\{\exp\left[-(1+\nu'')\frac{\alpha t}{m}\right]-1\right\}\Bigg)^{2/(1+\nu'')}. \tag{3.21}$$

From Eq. (3.19) one notices that $v_0(t)$ decays exponentially and does not depend on q; moreover, $G(t)$ and $Z(t)$ reduce to constants in the long-time limit, as expected, in such way to recover the stationary solution in Eq. (3.12). Similar results follow by applying the ansatz above for Eq. (3.5) with $F_x = -\gamma x$, from which one obtains the time-dependent solution of Refs. [7,8] [cf. Eq. (3.9)].

Herein we shall discuss two cases where the ansatz of Eq. (3.18) is applied for the NLKE of Eq. (3.2), involving the joint probability distribution $P(x,v,t)$, defined for an entropic index $q < 2$. In the first case we consider the external force $F_x = 0$ and $F_v = -\alpha v$, enabling us to verify

$$\frac{dx_0}{dt} = v_0, \qquad \frac{dv_0}{dt} = -\frac{\alpha}{m}v_0, \tag{3.22}$$

and

$$\begin{aligned}\frac{dG_{11}}{dt} &= 2G_{12},\\ \frac{dG_{12}}{dt} &= G_{22} - \frac{\alpha}{m}G_{12},\\ \frac{dG_{22}}{dt} &= -\frac{2\alpha}{m}G_{22} + 2(2-q)\frac{\Gamma}{m^2}Z^{q-1},\\ \frac{1}{Z}\frac{dZ}{dt} &= (2-q)\frac{\Gamma}{m^2}Z^{q-1}\frac{G_{11}}{\det G} - \frac{\alpha}{m}.\end{aligned} \tag{3.23}$$

In the second, we use an external harmonic force $F_x = -\gamma x$ and $F_v = -\alpha v$, which leads to the following system of equations:

$$\frac{dx_0}{dt} = v_0, \qquad \frac{dv_0}{dt} = -\frac{\alpha}{m}v_0 - \frac{\gamma}{m}x_0, \tag{3.24}$$

and

$$\begin{aligned}\frac{dG_{11}}{dt} &= 2G_{12},\\ \frac{dG_{12}}{dt} &= G_{22} - \frac{\alpha}{m}G_{12} - \frac{\gamma}{m}G_{11},\\ \frac{dG_{22}}{dt} &= -\frac{2\alpha}{m}G_{22} - \frac{2\gamma}{m}G_{12} + 2(2-q)\frac{\Gamma}{m^2}Z^{q-1},\\ \frac{1}{Z}\frac{dZ}{dt} &= (2-q)\frac{\Gamma}{m^2}Z^{q-1}\frac{G_{11}}{\det G} - \frac{\alpha}{m}.\end{aligned} \tag{3.25}$$

In both cases above, due the normalization requirement, the time-dependent coefficients are related by

$$Z(t)(\det G)^{-1/2} = (2-q), \tag{3.26}$$

where $\langle f(x,v)\rangle = \int f(x,v)P(x,v,t)dxdv$. Using this definition for average values, one verifies that

$$\langle x\rangle = x_0, \quad \langle v\rangle = v_0, \tag{3.27}$$

and

$$\langle (x_i - \langle x_i\rangle)(x_j - \langle x_j\rangle)\rangle = \frac{1}{2(3-2q)}G_{ij}. \tag{3.28}$$

Thus, these equations yield an interpretation for x_0, v_0, and G_{ij} in terms of the mean values of the position, velocity, and fluctuations.

The solutions of Eqs. (3.22) are given by

$$\begin{aligned}x_0(t) &= x_0(0) + \frac{mv_0(0)}{\alpha}\left[1 - \exp\left(-\frac{\alpha t}{m}\right)\right],\\ v_0(t) &= v_0(0)\exp\left(-\frac{\alpha t}{m}\right).\end{aligned} \tag{3.29}$$

On the other hand, the solutions of Eqs. (3.24) are precisely the well-known position and velocity for a damped harmonic oscillator. As pointed out, these solutions refer to the mean position and velocity described by Eq. (3.18).

The sets of Eqs. (3.23) and (3.25) provide information about the dispersion of the packet given by Eq. (3.18). One sees that in the case $q = 1$ these equations decouple, becoming much simpler; otherwise, they represent a set of coupled equations that do not present simple solutions. However, below we investigate the large-time regime, which yields some intuition about the problem.

C. Large-time limit solutions

Numerical investigations of Eqs. (3.25) indicate that the matrix components $G_{ij}(t)$ become constant for large t; this fact is consistent with the existence of the stationary solution of Eq. (3.14). On the other hand, numerical investigations of Eqs. (3.23) suggest an asymptotic power-law behavior for the matrix components $G_{ij}(t)$; therefore, we reduce our analytical investigation of Eqs. (3.23) in the limit $t \gg 1$ by using

$$\begin{aligned}G_{11}(t) &= B_{11}t^{\sigma_{11}},\\ G_{12}(t) &= B_{12}t^{\sigma_{12}},\\ G_{22}(t) &= B_{22}t^{\sigma_{22}},\\ Z(t) &= Bt^{\sigma}.\end{aligned} \tag{3.30}$$

We note that the matrix G is symmetric, so $G_{12} = G_{21}$, thus reducing the number of equations; moreover, one has the relation $G_{22}^{-1} = G_{11}/(\det G)$. Substituting the solutions proposed in Eq. (3.30) into the set of Eqs. (3.23), we obtain

$$\begin{aligned}G_{11}(t) &= B^{q-1}\frac{4\Gamma}{\alpha^2}\frac{(2-q)^2}{(3-q)}t^{(3-q)/(4-2q)},\\ G_{12}(t) &= B^{q-1}\frac{\Gamma}{\alpha^2}(2-q)t^{(q-1)/(4-2q)},\\ G_{22}(t) &= B^{q-1}\frac{\Gamma}{\alpha m}(2-q)t^{(q-1)/(4-2q)},\\ Z(t) &= Bt^{1/(4-2q)},\end{aligned} \tag{3.31}$$

where we made use of the normalization relation of Eq. (3.26). Since $q < 2$ in the present analysis, one sees that the coefficients G_{12} and G_{22} increase slower than G_{11}.

D. H-theorem for a limit case

Fokker-Planck equations, both in their linear [19,38] and nonlinear forms [11,32], can be written as continuity equations, so the stationary state is found by means of the assumption

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} J_i = 0, \tag{3.32}$$

where $\{J_i\}$ represent components of a N-dimensional probability-current-density vector. Hence, the probability current must be constant, and, furthermore, in order to satisfy the normalization constraint for all times, one can show that it must be zero at the equilibrium state. In contrast to this, in the case of the Kramers equation a zero normal component of the probability current density at the surface does not guarantee a null current inside the volume. Indeed, one can have circular and tangent currents, so a stationary state may occur, which does not necessarily correspond to an equilibrium state. This happens even in the linear case [38], and, as a consequence, it is not possible to prove the H-theorem by making use of the Kramers equation in its general form. Below we present the procedure used to prove the H-theorem and consider a limit for which this theorem can be achieved.

The H-theorem for a system subjected to forces $F_v = -dK(v)/dv$ and $F_x = -dV(x)/dx$, i.e., under a potential, $\Phi = V(x) + K(v)$, corresponds to a well-defined sign for the time derivative of the free-energy functional,

$$F = U[P] - \theta S[P], \tag{3.33}$$

where

$$U[P] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv [V(x) + K(v)] P(x,v,t), \tag{3.34}$$

with $S[P]$ representing an entropic functional and θ a temperature-like parameter.

Considering the general entropic form

$$S[P] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv g[P]; \quad g[0] = g[1] = 0;$$
$$\frac{d^2 g}{dP^2} \leqslant 0, \tag{3.35}$$

we can write

$$\frac{dF}{dt} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \left\{ [V(x) + K(v)] - \theta \frac{dg}{dP} \right\} \frac{\partial P}{\partial t}. \tag{3.36}$$

Substituting the NLKE of Eq. (3.2) for the time derivative in Eq. (3.36) and noting the physical requirements for probability normalization, i.e., $P(x,v,t)|_{x,v\to\pm\infty} = 0$, $(\partial P/\partial x)|_{x,v\to\pm\infty} = 0$, and $(\partial P/\partial v)|_{x,v\to\pm\infty} = 0$, we obtain

$$\frac{dF}{dt} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \bigg[\left(-F_v - \theta \frac{d^2 g}{dP^2} \frac{\partial P}{\partial v} \right) \times \frac{P}{m} \left(-F_v + \nu P^{\nu-2} \frac{\Gamma}{2m} \frac{\partial P}{\partial v} - F_x \right) + F_v \frac{P}{\alpha} \left(-F_x - \theta \frac{d^2 g}{dP^2} \frac{\partial P}{\partial x} \right) \bigg]. \tag{3.37}$$

From the equation above one may see that there is no choice for $g[P]$ that guarantees a well-defined sign for dF/dt; then we conclude that the stationary state of Eq. (3.14) is not necessarily an equilibrium state. However, if one sets $F_x = 0$, Eq. (3.37) can be written as

$$\frac{dF}{dt} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \bigg[\left(-F_v - \theta \frac{d^2 g}{dP^2} \frac{\partial P}{\partial v} \right) \times \frac{P}{m} \left(-F_v + \nu P^{\nu-2} \frac{\Gamma}{2m} \frac{\partial P}{\partial v} \right) \bigg]. \tag{3.38}$$

Considering $-\theta d^2 g/dP^2 = \nu \Gamma P^{\nu-2}/(2m)$, we obtain

$$\frac{dF}{dt} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \frac{P}{m} \left(-F_v + \nu P^{\nu-2} \frac{\Gamma}{2m} \frac{\partial P}{\partial v} \right)^2 \leqslant 0, \tag{3.39}$$

yielding the H-theorem for the NLFPE in Eq. (3.7). The above choice for the entropic functional $g[P]$ leads to Tsallis entropy [3,5,6],

$$S_\nu[P] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv g[P] = k \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \frac{P - P^\nu}{\nu - 1}, \tag{3.40}$$

where k is a constant with dimensions of entropy.

Following the above approach, a connection between the NLFPE associated with the probability $P(v,t)$ in Eq. (3.7) for velocities with nonextensive statistical mechanics can be shown, reinforcing the one found before by means of its stationary-state solution [cf. Eq. (3.12)]. Similar connections were found for the NLFPE associated with the probability $P(x,t)$ in Eq. (3.5) for positions (see, e.g., Refs. [27,28,30,32]). The fact that both limit equations, namely Eqs. (3.5) and (3.7), as well as the stationary-state solution for $P(x,v)$ in Eq. (3.14), are all related in some way to Tsallis entropy does strongly suggest an association of the Kramers equation of Eq. (3.2) with nonextensive statistical mechanics, even though a general proof of an H-theorem cannot be achieved from this equation.

In the next section we discuss previous measurements in a biological system, namely the motion of *Hydra* cells, within the context of the present NLKE.

IV. THE MOTION OF *HYDRA VIRIDISSIMA* CELLS

In this section we discuss experimental results related to the motion of *Hydra viridissima* cells within the present theoretical context of a NLKE. *Hydra viridissima* is a small organism which presents a cylindrical body with two layers of cells (the inner one, called the endoderm, and the outer one, called the ectoderm) and lives in dirty water [40,41]. The experiments described in Ref. [41] analyzed statistical properties associated with the two-dimensional motion of the centers of mass of single endodermal *Hydra* cells in two different aggregate types, endodermal and ectodermal. The velocity distributions for the two components v_x and v_y were well adjusted by similar q-Gaussian distributions with an index $\tilde{q} = 1.5 \pm 0.05$. Moreover, an anomalous diffusion (i.e., superdiffusive) was found in this two-dimensional space, characterized by $\langle r^2(t)\rangle \propto t^a$ with $a = 1.23 \pm 0.1$ [41].

In order to treat this system, we will deal with a two-dimensional Kramers equation defined for a probability distribution $P(x,y,v_x,v_y,t)$ instead of the one-dimensional case analyzed in the previous sections. Herein, we consider the following NLKE, which consists in a natural extension of Eq. (3.2):

$$\frac{\partial P}{\partial t} = \frac{\Gamma}{2m^2}\left(\frac{\partial^2 P^\nu}{\partial v_x^2} + \frac{\partial^2 P^\nu}{\partial v_y^2}\right) + \alpha\left(\frac{\partial(v_x P)}{\partial v_x} + \frac{\partial(v_y P)}{\partial v_y}\right) - \frac{\partial(v_x P)}{\partial x} - \frac{\partial(v_y P)}{\partial y}. \tag{4.1}$$

The equation proposed above is based on the experimental study of Ref. [41], from which we assume the two important conditions: (i) No external forces act on the cells, i.e., $F_x = F_y = 0$; and (ii) we use the same coefficient and exponent for both components of the nonlinear terms, since the observed velocity distributions yielded typically similar behavior for both components v_x and v_y. Moreover, we consider an ansatz, which is essentially an extension of the one used in the corresponding one-dimensional equation,

$$P(x,y,v_x,v_y,t) = \frac{1}{Z(t)} \exp_q\left\{\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}[x_i - x_{i0}(t)]G_{ij}^{-1}(t)[x_j - x_{j0}(t)]\right\}, \tag{4.2}$$

with $(x_1,x_2,x_3,x_4) \equiv (x,y,v_x,v_y)$ and $q = 2 - \nu$.

From the joint probability distribution $P(x,y,v_x,v_y,t)$ one can calculate reduced probabilities, as done in Eq. (3.17); for instance, one has

$$P(v_x,t) = \int P(x,y,v_x,v_y,t)dxdydv_y, \tag{4.3}$$

leading to

$$P(v_x,t) \propto \exp_{\tilde{q}}\{-\sigma(t)[v_x - \tilde{v}_{x0}(t)]^2\}. \tag{4.4}$$

In the distribution above, $\tilde{v}_{x0}(t)$ denotes the mean value, whereas $\sigma(t)$ represents a quantity related to the standard deviation of v_x at time t. Moreover, the index $\tilde{q}$ results from three successive applications of Eq. (3.16) (one operation for each integration), yielding the following relation with the index q of the ansatz in Eq. (4.2):

$$\tilde{q} = \frac{3-q}{5-3q}. \tag{4.5}$$

Consistently with our previous study of the one-dimensional case, $\sigma(t) \to$ const. and $\tilde{v}_{x0} \to 0$ for large t. Additionally, due to the symmetry $(x,v_x) \leftrightarrow (y,v_y)$ of Eq. (4.1), similar results hold for v_y.

Based on the experimental estimate [41], $\tilde{q} = 1.5 \pm 0.05$, for both distributions $P(v_x,t)$ and $P(v_y,t)$, we will consider herein $\tilde{q} = 3/2$ for simplifying the calculations that follow. From Eq. (4.5) one obtains $q = 9/7$ as the index associated with the distribution $P(x,y,v_x,v_y,t)$, leading to the exponent of Eq. (4.1), $\nu = 2 - q = 5/7$. Now, let us analyze the mean-square displacement

$$\langle r^2(t)\rangle = \langle[x(t) - x_0(t)]^2\rangle + \langle[y(t) - y_0(t)]^2\rangle, \tag{4.6}$$

which, in principle, cannot be calculated analytically, since we do not know the precise form of $P(x,y,v_x,v_y,t)$. However, within the context of the present NLKE, it appears natural to assume that the reduced distribution $P(x,t)$, characterized by an index $\tilde{q} = 3/2$, satisfies the NLFPE of Eq. (3.5) with $\nu' = 2 - \tilde{q} = 1/2$. Considering this assumption, a dimensional analysis leads to $\langle[x(t) - x_0(t)]^2\rangle \propto t^{2/(3-\tilde{q})}$ [5], with a similar behavior for the the y-component contribution. Hence, one obtains

$$\langle r^2(t)\rangle \propto t^a \quad (a = 4/3), \tag{4.7}$$

which agrees with the experimental value of Ref. [41], $a = 1.23 \pm 0.1$, within the error bars.

The close agreement of the superdiffusion exponent is quite astonishing and support the assumptions considered in the above analysis. Therefore, we have shown that Eq. (4.1), which represents a two-dimensional extension of the one-dimensional NLKE studied herein, yields reduced q-Gaussian distributions for both velocity components v_x and v_y, as well as for both position components x and y, which are consistent with experimental observations for the motion of *Hydra viridissima* cells.

V. CONCLUSIONS

We have investigated a nonlinear Kramers equation for a joint probability distribution $P(x,v,t)$, related to nonextensive statistical mechanics. The stationary state, as well as time-dependent solutions, were analyzed analytically. In the time-dependent case, we have found an associated set of equations and solved it analytically in the asymptotic regime, showing that the stationary-state solutions are recovered in the long-time limit. We have calculated the corresponding nonlinear Fokker-Planck equations for the marginal probabilities $P(x,t)$ and $P(v,t)$, which may present, in principle, different degrees of nonlinearity when compared with the original Kramers equation. We have also discussed the proof of the H-theorem by making use of the nonlinear Kramers equation; it is shown that this proof is not possible in general (even in the linear case). However, in a particular case of a null position-dependent force, $F_x = -[dV(x)/dx] = 0$, this theorem is achieved.

Within this context, an application was considered, where we have compared our theoretical results with measurements for the motion of *Hydra viridissima* cells on the surface of dirty water. By analyzing reduced probabilities derived from $P(x,y,v_x,v_y,t)$, satisfying a two-dimensional nonlinear Kramers equation that represents an extension of the one-dimensional equation studied herein, we have shown that reduced q-Gaussian distributions for both velocity components v_x and v_y, as well as those for both position components x and y, are consistent with the experimental observations. Particularly, the anomalous-diffusion exponent obtained theoretically agrees with the observed one, taking into account the experimental error bars. The good agreement between the present theoretical approach with observations of an anomalous diffusion in a biological system suggests that the nonlinear Kramers equation investigated herein should be useful for describing other complex systems in nature as well.

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