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q-moments remove the degeneracy associated with the inversion of the q-Fourier transform

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Abstract. It was recently proven (Hilhorst 2010 *J. Stat. Mech.* P10023) that the q-generalization of the Fourier transform is not invertible in the full space of probability density functions for $q > 1$. It has also been recently shown that this complication disappears if we dispose of the q-Fourier transform not only of the function itself, but also of all of its shifts (Jauregui and Tsallis 2011 *Phys. Lett.* A **375** 2085). Here we show that another route exists for completely removing the degeneracy associated with the inversion of the q-Fourier transform of a given probability density function. Indeed, it is possible to determine this density if we dispose of some extra information related to its q-moments.

Keywords: rigorous results in statistical mechanics, exact results, diffusion

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1. Introduction

Nonextensive statistical mechanics [1], a current generalization of the Boltzmann–Gibbs theory, is actively studied in diverse areas of physics and other sciences [2, 3]. This theory is based on a nonadditive entropy, commonly denoted by S_q, that depends, in addition to the probabilities of the microstates, on a real parameter q, which is inherent to the system and makes S_q extensive. In the limit $q \to 1$, nonextensive statistical mechanics yields the Boltzmann–Gibbs theory. This new theory has successfully described many physical and computational experiments. Such systems typically are nonergodic ones, with long-range interactions, long memory and/or other nontrivial ingredients: see, for example, [4]–[12].

The development of nonextensive statistical mechanics introduced, in addition to the generalization of some physical concepts like the Boltzmann–Gibbs–Shannon–von Neumann entropy, the generalization of some mathematical concepts. Remarkable ones are the generalizations of the classical central limit theorem and the Lévy–Gnedenko one. These extensions are based on a generalization of the Fourier transform (FT), namely the q-Fourier transform (q-FT) [13, 14]. These generalized theorems respectively establish, for $q > 1$, q-Gaussians and (q, α)-stable distributions as attractors when the considered random variables are correlated in a special manner.

If $1 < q < 3$, a q-Gaussian is a generalization of a Gaussian defined as a function $G_{q,\beta} : \mathcal{R} \to \mathcal{R}$ such that

$$G_{q,\beta}(x) = \frac{\sqrt{\beta}}{C_q[1 + (q-1)\beta x^2]^{1/(q-1)}} \equiv \frac{\sqrt{\beta}}{C_q} \exp_q(-\beta x^2), \tag{1}$$

where $\beta > 0$ and C_q is a normalization constant given by

$$C_q = \frac{\sqrt{\pi}\Gamma((3-q)/2(q-1))}{\sqrt{q-1}\Gamma(1/(q-1))}. \tag{2}$$

A q-Gaussian is not normalizable for $q \geq 3$. Its variance is finite for $q < 5/3$; above this value, it diverges. When correlations can be neglected, $q \to 1$ and $G_{q,\beta}(x) \to (\beta/\pi)^{1/2} \exp(-\beta x^2)$, which is a Gaussian.

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The q-FT of a non-negative measurable function f, denoted by $F_q[f]$, is defined, for $1 \leq q < 3$, as

$$F_q[f](\xi) = \int_{\mathrm{supp}\, f} f(x) \exp_q(\mathrm{i}\xi x[f(x)]^{q-1})\,\mathrm{d}x, \tag{3}$$

where $\mathrm{supp}\, f$ stands for the support of f, and $\exp_q(\mathrm{i}x) = \mathrm{pv}\,[1 + (1-q)\mathrm{i}x]^{1/(1-q)}$ for any real number x, pv being the notation for principal value. This is a nonlinear integral transform when $q > 1$. Its relevance in [13] is that it transforms a q-Gaussian into another one. Hence the q-FT is invertible in the space of q-Gaussians [15]. However, it was recently proven, by means of counterexamples, that the q-FT is not invertible in the full space of probability density functions (pdf's) [16]. In connection with this problem, it is worth mentioning that it has been found an interesting property of the q-FT which enables the determination of a given pdf from the knowledge of the q-FT of an arbitrary translation of such pdf's [17].

Here we will discuss the counterexamples given in [16], and we will show that it is possible to determine the pdf's considered in the counterexamples from the knowledge of their q-FT and some extra information related with their q-moments, defined here below.

Let Q be a real number and f be a pdf of some random variable X such that the quantity

$$\nu_Q[f] = \int_{\mathrm{supp}\, f} [f(x)]^Q\,\mathrm{d}x \tag{4}$$

is finite. Then, we can define an escort pdf [18] for X, denoted by f_Q, as follows:

$$f_Q(x) = \frac{[f(x)]^Q}{\nu_Q[f]}. \tag{5}$$

The moments of f_Q, which are called Q-moments of f, are given by

$$\Pi_Q^{(n)}[f] = \int_{\mathrm{supp}\, f} x^n f_Q(x)\,\mathrm{d}x = \frac{\mu_Q^{(n)}[f]}{\nu_Q[f]}, \tag{6}$$

where $\mu_Q^{(n)}[f]$ is the unnormalized nth Q-moment of f, defined as follows:

$$\mu_Q^{(n)}[f] = \int_{\mathrm{supp}\, f} x^n [f(x)]^Q\,\mathrm{d}x, \tag{7}$$

n being a positive integer.

The characteristic function of X is basically given by the Fourier transform of f, $F[f]$. It is well known that all the moments of f can be obtained from the successive derivatives of the characteristic function of X at the origin. It was shown that the successive derivatives of the q-FT of f at the origin are related to specific unnormalized Q-moments of f by the following equation [19]:

$$\left.\frac{\mathrm{d}^n F_q[f](\xi)}{\mathrm{d}\xi^n}\right|_{\xi=0} = \mathrm{i}^n \left\{\prod_{j=0}^{n-1} [1 + j(q-1)]\right\} \mu_{q_n}^{(n)}[f], \tag{8}$$

where $q_n = nq - (n-1)$. We can see from this relation that, if the q-FT of f does not depend on a certain parameter that appears in f, then the unnormalized nth q_n-moments

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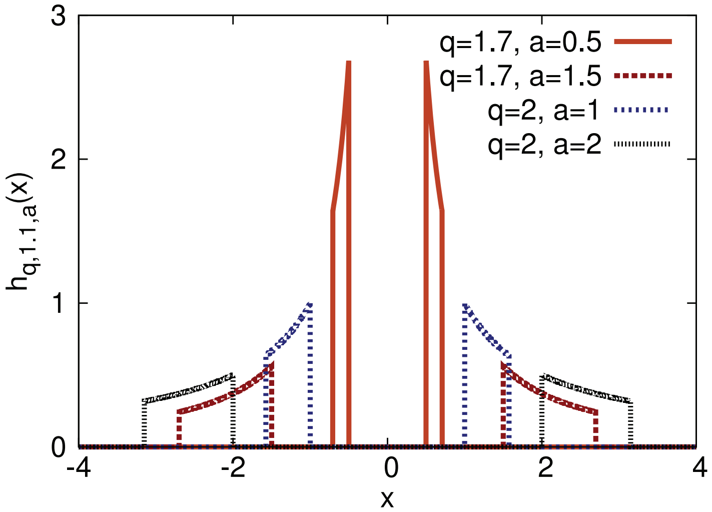

Figure 1. Representation of $h_{q,\lambda,a}$ for $\lambda = 1.1$ and different values of q and a.

also do not depend on such a parameter. Therefore, these unnormalized moments are unable to identify the pdf f from its q-FT. As will soon become clear, this difficulty does not exist for the set of $\{\nu_q\}$, which will then provide the desired identification procedure.

2. Hilhorst's examples

We discuss in this section two examples proposed by Hilhorst [16], where the pdf depends on a certain real parameter, which disappears when we take its q-FT. Therefore, at the step of looking at the inverse q-FT, we face an infinite degeneracy. Next we illustrate, in both examples, how the degeneracy is removed through the values of the $\{\nu_q\}$.

2.1. First example

Let us consider the function $h_{q,\lambda,a} : \mathcal{R} \to \mathcal{R}$ such that [16]

$$h_{q,\lambda,a}(x) = \left(\frac{\lambda}{|x|}\right)^{1/(q-1)} \tag{9}$$

if $a < |x| < b$, where $q > 1$, and (a, b, λ) are positive real numbers; otherwise $h_{q,\lambda,a}(x) = 0$ (see figure 1). We can impose the following normalization condition for this function:

$$\int_{-\infty}^{+\infty} h_{q,\lambda,a}(x)\,\mathrm{d}x = 1. \tag{10}$$

From this, it follows that one parameter among q, λ, a, b depends on the other ones. Choosing b as the dependent parameter, we get

$$b = \left[\frac{q-2}{2(q-1)}\lambda^{1/(1-q)} + a^{(q-2)/(q-1)}\right]^{(q-1)/(q-2)} \qquad q \neq 2 \tag{11a}$$

$$= a\mathrm{e}^{1/2\lambda} \qquad q = 2. \tag{11b}$$

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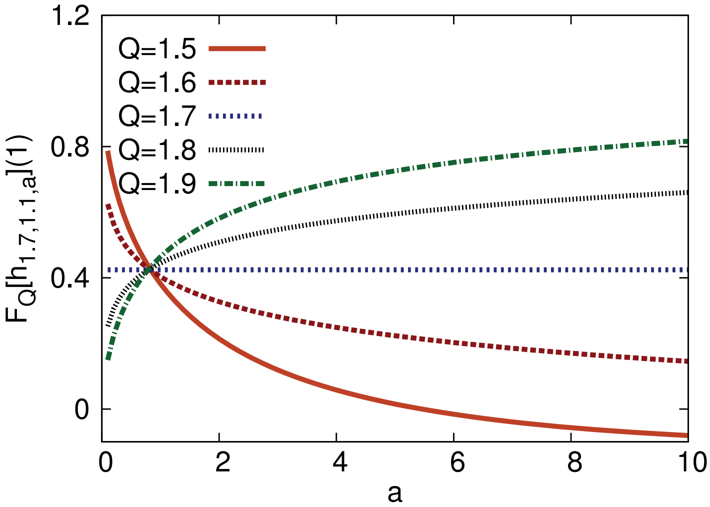

Figure 2. The dependence on a of $F_Q[h_{1.7,1.1,a}](1)$ for different values of Q.

Given Q such that $1 \le Q < 3$, the Q-FT of $h_{q,\lambda,a}$ can be easily reduced to the following expression:

$$F_Q[h_{q,\lambda,a}](\xi) = 2\int_a^b \left(\frac{\lambda}{x}\right)^{1/(q-1)} \cos_Q\left(\xi x \left(\frac{\lambda}{x}\right)^{(Q-1)/(q-1)}\right) \mathrm{d}x, \qquad (12)$$

where $\cos_q$ is the q-generalization of the trigonometric function cos which is defined by [20] $\cos_q x = \Re(\exp_q(\mathrm{i}x))$. When $q \neq 1$, we have that

$$\cos_q x = [1 + (1-q)^2 x^2]^{1/2(1-q)} \cos\left(\frac{\arctan((1-q)x)}{1-q}\right). \qquad (13)$$

It is easy to notice from (12) that the Q-FT of $h_{q,\lambda,a}$ depends on a if $Q \neq q$. However, it does not depend on a when $Q = q$ (see figure 2), when it is given by $F_q[h_{q,\lambda,a}](\xi) = \cos_q(\xi\lambda)$.

Consequently, there exist infinite functions $h_{q,\lambda,a}$ with the same q and λ but different a, which have the same q-FT. Therefore, it is not possible to determine $h_{q,\lambda,a}$ just from the knowledge of its q-FT. However, it may be possible to obtain $h_{q,\lambda,a}$ from its q-FT and some extra information. For example, we would be able to determine $h_{q,\lambda,a}$ if we knew the q-FT of an arbitrary translation of $h_{q,\lambda,a}$ [17]. Here we will give another approach to this problem.

As $h_{q,\lambda,a}$ is a non-negative function, which obeys the normalization condition (10), it can be interpreted as a pdf of some random variable. Moreover, for any real number Q, we have that

$$\nu_Q[h_{q,\lambda,a}] = 2\lambda^{Q/(q-1)}[b^{1-Q/(q-1)} - a^{1-Q/(q-1)}]\frac{(q-1)}{q-1-Q} \qquad Q \neq q-1 \qquad (14a)$$

$$= 2\lambda \ln(b/a) \qquad Q = q-1 \qquad (14b)$$

is finite. With n being an even positive integer, we have also that the unnormalized nth Q-moment of $h_{q,\lambda,a}$ is given by

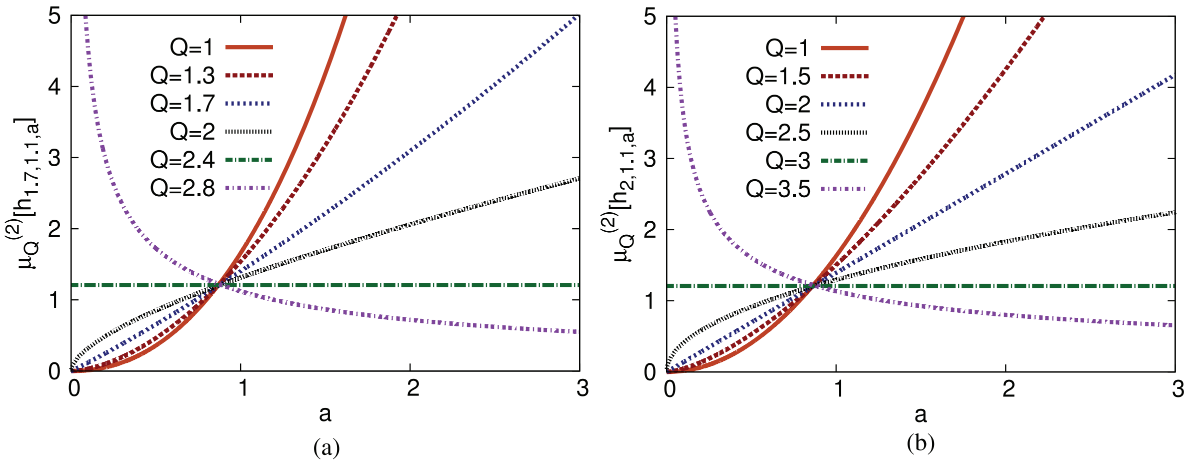

Figure 3. The dependence on a of the quantities (a) $\mu_Q^{(2)}[h_{1.7,1.1,a}]$ and (b) $\mu_Q^{(2)}[h_{2,1.1,a}]$ for different values of Q.

$$\mu_Q^{(n)}[h_{q,\lambda,a}] = 2\lambda^{Q/(q-1)}[b^{n+1-Q/(q-1)} - a^{n+1-Q/(q-1)}]\frac{(q-1)}{(n+1)(q-1)-Q} \qquad Q \neq (n+1)(q-1) \tag{15a}$$

$$= 2\lambda^{n+1}\ln(b/a) \qquad Q = (n+1)(q-1). \tag{15b}$$

Then, finally, the nth Q-moment of $h_{q,\lambda,a}$ is given by

$$\Pi_Q^{(n)}[h_{q,\lambda,a}] = \frac{b^n - a^n}{n\ln(b/a)} \qquad Q = q-1 \tag{16a}$$

$$= \frac{na^n b^n}{b^n - a^n}\ln(b/a) \qquad Q = (n+1)(q-1) \tag{16b}$$

$$= \left[\frac{b^{n+1-Q/(q-1)} - a^{n+1-Q/(q-1)}}{b^{1-Q/(q-1)} - a^{1-Q/(q-1)}}\right]\frac{(q-1-Q)}{(n+1)(q-1)-Q} \qquad \text{otherwise.} \tag{16c}$$

It is clear that $\mu_Q^{(m)}[h_{q,\lambda,a}] = 0$ and $\Pi_Q^{(m)}[h_{q,\lambda,a}] = 0$ for any odd positive integer m, since $h_{q,\lambda,a}(x)$ is an even function.

As the q-FT of $h_{q,\lambda,a}$ does not depend on a, then, according to (8), the nth q_n-moment of $h_{q,\lambda,a}$ does not depend on a either, where $q_n = nq-(n-1)$. In fact, if $q \neq 2$, we have that

$$\mu_{q_n}^{(n)}[h_{q,\lambda,a}] = \frac{2(q-1)}{q-2}\lambda^{n+1/(q-1)}[b^{(q-2)/(q-1)} - a^{(q-2)/(q-1)}]. \tag{17}$$

Then, using (11a), we obtain that $\mu_{q_n}^{(n)}[h_{q,\lambda,a}] = \lambda^n$. If $q = 2$, we have that $\mu_{n+1}^{(n)}[h_{q,\lambda,a}] = 2\lambda^{n+1}\ln(b/a)$ and, using (11b), we obtain that $\mu_{n+1}^{(n)}[h_{q,\lambda,a}] = \lambda^n$.

While the unnormalized Q-moments of $h_{q,\lambda,a}$ may not depend on a (see figure 3), we can straightforwardly verify from (14a) and (14b) that the quantity $\nu_Q[h_{q,\lambda,a}]$ depends

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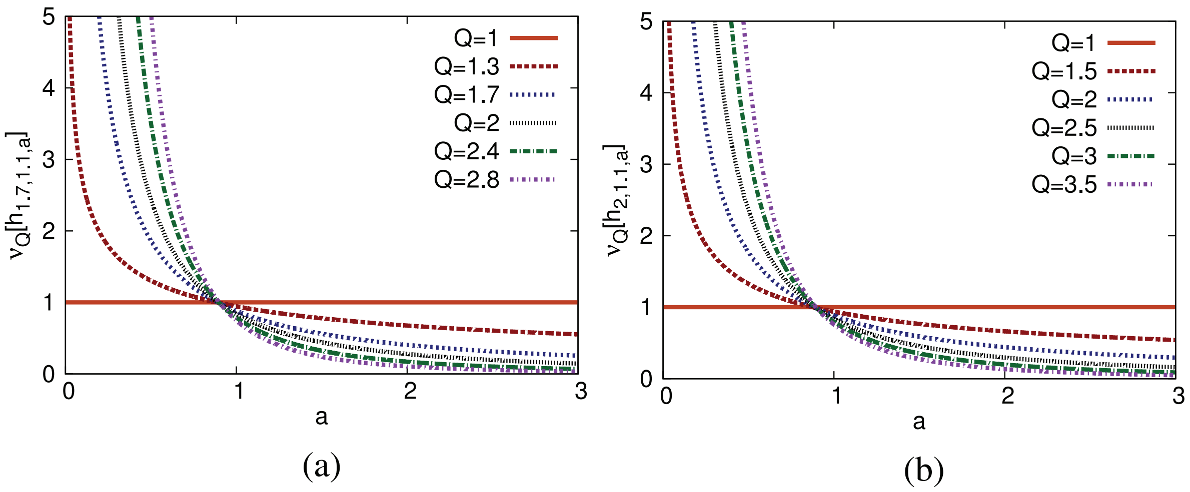

Figure 4. The dependence on a of the quantities (a) $\nu_Q[h_{1.7,1.1,a}]$ and (b) $\nu_Q[h_{2,1.1,a}]$ for different values of Q.

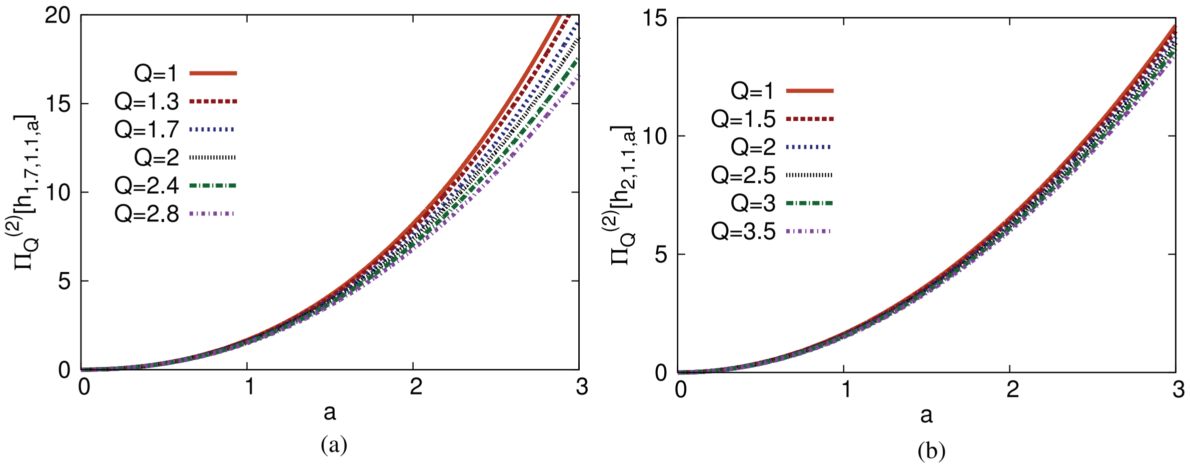

Figure 5. The dependence on a of the quantities (a) $\Pi_Q^{(2)}[h_{1.7,1.1,a}]$ and (b) $\Pi_Q^{(2)}[h_{2,1.1,a}]$ for different values of Q.

monotonically on a for any $Q \neq 1$ (see figure 4). The same is true for the normalized Q-moments (see figure 5). Hence, the knowledge of the q-FT of $h_{q,\lambda,a}$ and the value of some $\nu_Q[h_{q,\lambda,a}]$ with $Q \neq 1$ (extra information) is sufficient to determine the pdf $h_{q,\lambda,a}$. We should notice that $\nu_1[h_{q,\lambda,a}] = 1$ (it does not depend on a), then the extra information in this case is trivial.

2.2. Second example

Let us consider now the function $f_{q,A} : \mathcal{R} \to \mathcal{R}$ such that [16]

$$f_{q,A}(x) = \frac{[\alpha_{q,A}(x)]^{1/(1-q)}}{C_q\{1 + (q-1)x^2[\alpha_{q,A}(x)]^{-2}\}^{1/(q-1)}} \tag{18}$$

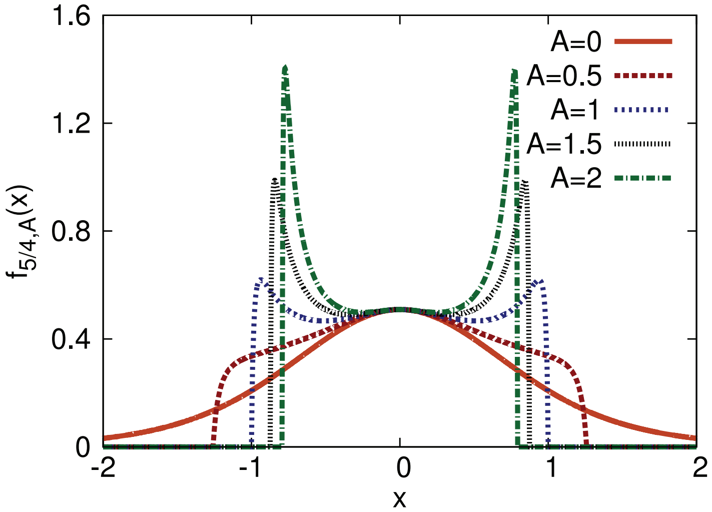

Figure 6. Representation of $f_{5/4,A}$ for different values of A.

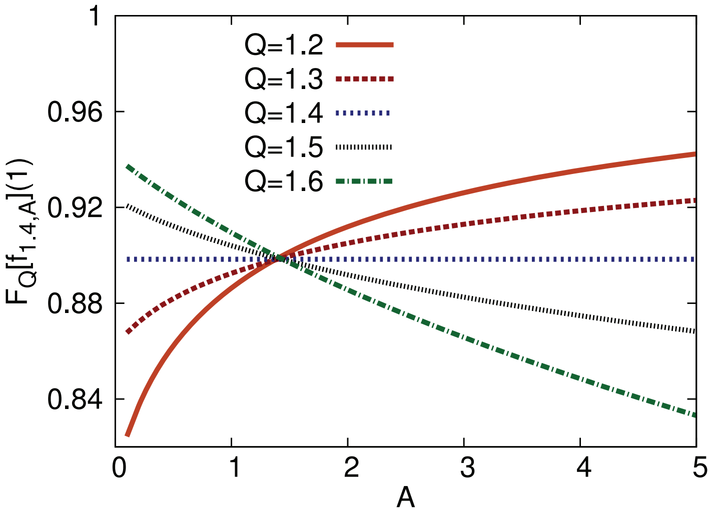

Figure 7. The dependence on A of $F_Q[f_{1.4,A}](1)$ for different values of Q.

if $|x|^{(q-2)/(q-1)} > A$, where $1 < q < 2$, $A \geq 0$:

$$\alpha_{q,A}(x) = [1 - A|x|^{(2-q)/(q-1)}]^{(q-1)/(2-q)}, \tag{19}$$

and C_q is the normalization constant of a q-Gaussian given by (2); otherwise $f_{q,A}(x) = 0$ (see figure 6). We can easily notice that $f_{q,0}(x) = G_{q,1}(x)$, where $G_{q,\beta}(x)$ is defined in (1).

Let $1 < Q < 3$ and $A > 0$. The Q-FT of $f_{q,A}$ is given by (see figure 7)

$$F_Q[f_{q,A}](\xi) = \int_{-A^{(q-1)/(q-2)}}^{A^{(q-1)/(q-2)}} f_{q,A}(x) \exp_Q(\mathrm{i}\xi x[f_{q,A}(x)]^{Q-1})\,\mathrm{d}x. \tag{20}$$

In order to compute this integral in the particular case $Q = q$, we should notice first that

$$\begin{aligned}
\exp_q(\mathrm{i}\xi x[f_{q,A}(x)]^{q-1}) &= \exp_q\left(\frac{\mathrm{i}\xi x[\alpha_{q,A}(x)]^{-1}}{C_q^{q-1}\{1+(q-1)x^2[\alpha_{q,A}(x)]^{-2}\}}\right)\\
&= \mathrm{pv}\,\{1+(q-1)x^2[\alpha_{q,A}(x)]^{-2}\}^{1/(q-1)}\\
&\quad\times\left\{1+(1-q)\left\{\frac{-x^2}{[\alpha_{q,A}(x)]^2}+\frac{\mathrm{i}C_q^{1-q}\xi x}{\alpha_{q,A}(x)}\right\}\right\}^{1/(1-q)}\\
&= \{1+(q-1)x^2[\alpha_{q,A}(x)]^{-2}\}^{1/(q-1)}\exp_q\left(\frac{-x^2}{[\alpha_{q,A}(x)]^2}+\frac{\mathrm{i}C_q^{1-q}\xi x}{\alpha_{q,A}(x)}\right).
\end{aligned}\tag{21}$$

Then

$$\begin{aligned}
F_q[f_{q,A}](\xi) &= \frac{1}{C_q}\int_{-A^{(q-1)/(q-2)}}^{A^{(q-1)/(q-2)}}\frac{1}{[\alpha_{q,A}(x)]^{1/(q-1)}}\exp_q\left(\frac{-x^2}{[\alpha_{q,A}(x)]^2}+\frac{\mathrm{i}C_q^{1-q}\xi x}{\alpha_{q,A}(x)}\right)\mathrm{d}x\\
&= \frac{1}{C_q}\int_{-A^{(q-1)/(q-2)}}^{A^{(q-1)/(q-2)}}\frac{1}{[\alpha_{q,A}(x)]^{1/(q-1)}}\\
&\quad\times\ \exp_q\left(-\left[\frac{x}{\alpha_{q,A}(x)}-\frac{\mathrm{i}C_q^{1-q}\xi}{2}\right]^2-\frac{C_q^{2(1-q)}\xi^2}{4}\right)\mathrm{d}x.
\end{aligned}\tag{22}$$

Finally, using the change of variables

$$y = \frac{x}{\alpha_{q,A}(x)} - \frac{\mathrm{i}C_q^{1-q}\xi}{2},\tag{23}$$

we obtain that

$$F_q[f_{q,A}](\xi) = \frac{1}{C_q}\int_{-\infty-\mathrm{i}C_q^{1-q}\xi/2}^{+\infty-\mathrm{i}C_q^{1-q}\xi/2}\exp_q\left(-y^2-\frac{C_q^{2(1-q)}\xi^2}{4}\right)\mathrm{d}y,\tag{24}$$

which does not depend on A. Moreover, the RHS of (24) is equal to the q-FT of the q-Gaussian $G_{q,1}$ (see details in [13]), which, naturally, does not depend on A. Then, the knowledge of only the q-FT of $f_{q,A}$ would not be sufficient information to determine $f_{q,A}$. Hence, as in the first example, extra information is needed.

Let Q be a real number. Considering $f_{q,A}$ as a pdf of some random variable, we have that

$$\begin{aligned}
\nu_Q[f_{q,A}] &= \int_{-A^{(q-1)/(q-2)}}^{A^{(q-1)/(q-2)}}\frac{[\alpha_{q,A}(x)]^{Q/(1-q)}}{C_q^Q\{1+(q-1)x^2[\alpha_{q,A}(x)]^{-2}\}^{Q/(q-1)}}\mathrm{d}x\\
&= \frac{1}{C_q^Q}\int_{-A^{(q-1)/(q-2)}}^{A^{(q-1)/(q-2)}}\frac{1}{[\alpha_{q,A}(x)]^{Q/(q-1)}}\left[\exp_q\left(-\frac{x^2}{[\alpha_{q,A}(x)]^2}\right)\right]^Q\mathrm{d}x,
\end{aligned}\tag{25}$$

which is finite and depends on A when $Q \neq 1$ (see figure 8). The unnormalized nth Q-moment of $f_{q,A}$ for any positive integer n is given by

$$\mu_Q^{(n)}[f_{q,A}] = \frac{1}{C_q^Q}\int_{-A^{(q-1)/(q-2)}}^{A^{(q-1)/(q-2)}}\frac{x^n}{[\alpha_{q,A}(x)]^{Q/(q-1)}}\left[\exp_q\left(-\frac{x^2}{[\alpha_{q,A}(x)]^2}\right)\right]^Q\mathrm{d}x,\tag{26}$$

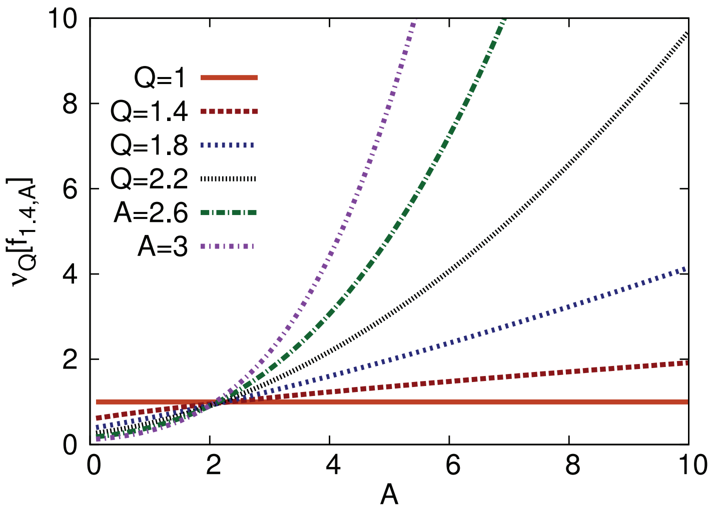

Figure 8. The dependence on A of the quantity $\nu_Q[f_{1.4,A}]$ for different values of Q.

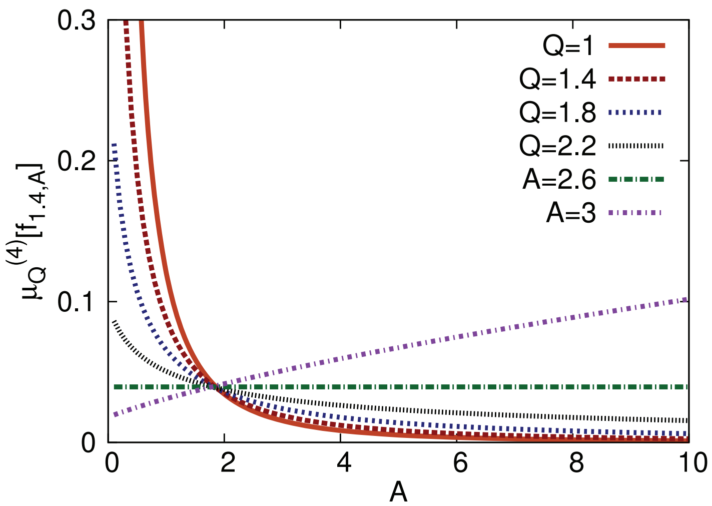

Figure 9. The dependence on A of the unnormalized fourth Q-moment of $f_{1.4,A}$ for different values of Q.

which depends on A except when $Q = q_n = nq - (n-1)$ (see figure 9). In this case, using the change of variables $y = x/\alpha_{q,A}(x)$, we obtain that

$$\mu_{q_n}^{(n)}[f_{q,A}] = \int_{-\infty}^{+\infty} y^n \left[\frac{1}{C_q}\exp_q(-y^2)\right]^{nq-(n-1)} \mathrm{d}y, \tag{27}$$

which is equal to the unnormalized nth q_n-moment of the q-Gaussian $G_{q,1}$. Therefore, we see that, as in the first example, the knowledge of any $\nu_Q[f_{q,A}]$ with $Q \neq 1$ enables the determination of the pdf $f_{q,A}$ from its q-FT.

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3. Conclusions

Both functions $h_{q,\lambda,a}$ and $f_{q,A}$ show that the q-FT is not invertible in the full space of pdf's, since their q-FT's do not depend on a and A, respectively. However, if $Q \neq q$, this problem would not occur for the Q-FT of both functions (see figures 2 and 7). In other words, the Q-FT of both functions with $Q \neq q$ would, in principle, be invertible. Furthermore, in the case $Q = q$, figures 4 and 8 show that the quantities $\nu_Q[h_{q,\lambda,a}]$ and $\nu_Q[f_{q,A}]$ depend monotonically on a and A, respectively, which removes the degeneracy. Therefore, the knowledge of the q-FT of both functions and a single value of $\nu_Q[h_{q,\lambda,a}]$ and $\nu_Q[f_{q,A}]$ is sufficient to determine the functions $h_{q,\lambda,a}$ and $f_{q,A}$.

If we were in the case that a pdf f depends on two or more parameters and its q-FT does not depend on more than one such parameter, we would expect this method of identification of the inverse q-FT to work as well as in the case of the functions considered in this paper. However, it might be possible that more than one value of ν_Q is needed.

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