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Iterative decomposition of Barabasi–Albert scale-free networks

S.S.B. Jácome [a,b,*], L.R. da Silva [b], A.A. Moreira [a], J.S. Andrade Jr. [a], H.J. Herrmann [a,c]

[a] Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil

[b] International Center for Complex Systems and Departamento de Física Teórica e Experimental, Universidade Federal do Rio Grande do Norte, Natal, Brazil

[c] Computational Physics, IfB, ETH-Hönggerberg, Schafmattstrasse 6, 8093 Zürich, Switzerland

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ABSTRACT

We study a decomposition process where all nodes with a targeted degree are removed from the network. Each removal step results in changes in the degrees of the remaining nodes, and other nodes may attain the targeted degree. The processes continue iteratively until no more nodes with the targeted degree are present in the decomposed network. The network model used in our study is the well known Barabasi–Albert network, that is built with an iterative growth based on preferential attachment. Our results show an exponential decay of the number of nodes removed at each step. The total number of nodes removed in the whole process depends on the targeted degree and decay with a power law controlled by the same exponent as the degree distribution of the network.

1. Introduction

One of the most common properties of real world networks is a power-law shape for the degree distribution, $P(k) \sim k^{-\gamma}$, with an exponent gamma typically in the range $2 < \gamma < 3$ [1]. Due to this highly skewed distribution, many processes that take place on these networks are more influenced by the hub nodes with extreme degrees than by the typical nodes of average degree [2]. For this reason such networks are called scale-free networks. The preferential attachment scheme, proposed by Barabasi and Albert [3] is a method by which one can build networks that naturally grow to obtain the scale-free characteristic. In this method nodes are introduced to the network one by one. The network starts with $m + 1$ nodes fully connected to each other. At each step a new node is added and makes m new connections to the nodes already present in the network. The parameter m is called here the aggregation coefficient. The key ingredient to obtain a scale-free distribution is the preferential attachment where the connections are directed with higher probability to the nodes with higher degree. Specifically, the probability of a node to receive a new connection is directly proportional to its degree. Many properties of the Barabasi–Albert (BA) networks have been studied, including random and targeted attack [4], extremum degree statistics [5], and cluster growth phenomena [6].

2. Model

In this paper we investigate a method of deconstructing BA networks. Our iterative decomposition process is performed by removing in successive steps all the nodes of a network that have a certain targeted degree K. At the initial step we identify all the nodes with degree K and remove from the network these nodes and their connections. This removal changes the degrees of the remaining nodes, and some of the nodes, that had more than K connections before, might have their degree changed to the targeted degree. As shown in Fig. 1, all the nodes that now have degree K are removed in the following step,

* Corresponding author at: Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil.
E-mail address: samyrbr@gmail.com (S.S.B. Jácome).

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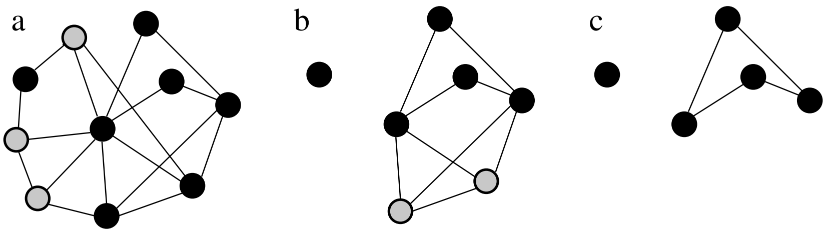

Fig. 1. Schematic illustration of the iterative decomposition process. In (a) we show a network with $N = 10$ nodes. The decomposition process starts by removing all the nodes with targeted degree $K = 3$, shown in gray. In (b) we show the network after the first removal step. Two other nodes now have $k = 3$, and will be removed. In (c) no nodes with degree $k = 3$ are present and we reach the final configuration of the iterative decomposition process.

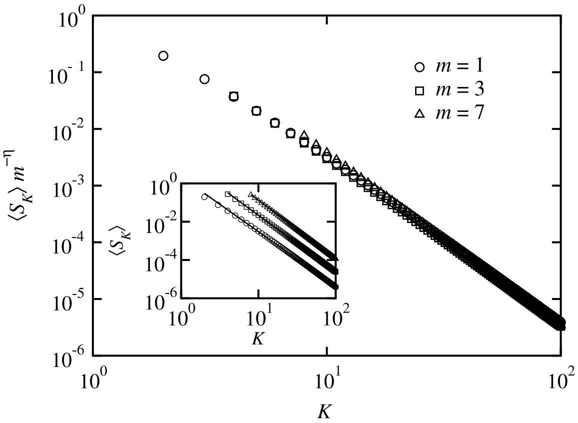

Fig. 2. Fraction of removed nodes at the end of the iterative process $\langle S_K \rangle$ as a function of the targeted degree K. The decomposition is performed on a Barabási–Albert (BA) network with $N = 2^{23}$ nodes built with different aggregation coefficients m. We can see a power-law decay with slope close to 3.0, independent of m. The data for different values of m can be collapsed by multiplying by a factor $m^{-\eta}$, with $\eta = 1.8$, as seen in the main panel. In the inset we show the original data.

and the process continues until we reach a network without any node with the targeted degree. This iterative decomposition method bears some resemblance, but is different from the k-core decomposition [7–9] where all the nodes with degree smaller than a certain value k are successively removed.

3. Results and discussion

The fraction of nodes removed at step t is given by $s_K(t)$, and the total fraction of removed nodes S_K is given by

$$S_K = \sum_{t=1}^{T_k} s_K(t), \tag{1}$$

where T_K is the number of steps when the end of the decomposition process is reached.

Similar to what is observed in the k-core decomposition [8–11], the total fraction of nodes removed from the Barabási–Albert (BA) network follows a power-law decay

$$\langle S_K \rangle = C_S K^{-\alpha} m^{-\eta}, \tag{2}$$

with $C_S \approx 3.76$, $\eta \approx 1.8$ and $\alpha \approx 3$. The exponent α is independent on the aggregation coefficient m and on the network size N, as seen in Fig. 2. One can note that $\langle S_K \rangle$ decays with the same exponent as the degree distribution. The curves for the fraction $\langle S_K \rangle$ can be collapsed by rescaling by a factor m^{η}. A similar rescaling with the parameter m can also be observed for the degree distribution, however with an slightly different exponent [12].

Fig. 3 shows that the average fraction of nodes removed at each step decays exponentially

$$\langle s_K(t) \rangle = C_K e^{-\lambda_K t}. \tag{3}$$

Although we present results for $m = 3$, similar behavior is observed for any value of m. The parameters λ_K and C_K give the characteristic decay and the initially removed fraction respectively, and are both dependent on the targeted degree K. In Fig. 4 we show the dependence of λ_K on the targeted degree and on the parameter m. We observe that λ_k displays a logarithmic growth,

$$\lambda_K = \delta \ln(C_\lambda K m^{-\nu}), \tag{4}$$

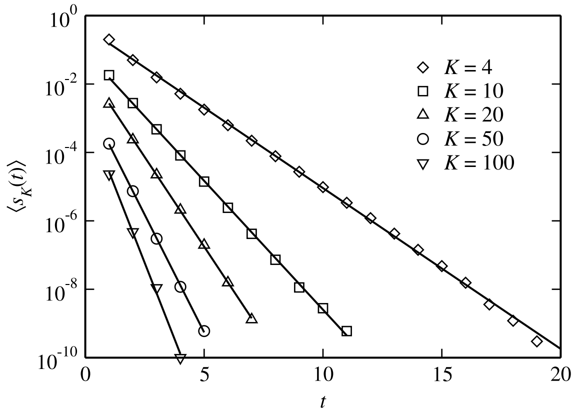

Fig. 3. The fraction of nodes removed at each step of the iterative process $\langle s_K(t) \rangle$. The iterative decomposition process was applied to BA networks with $m = 3$ and $N = 2^{25}$. Each curve is obtained for a different value of the targeted degree K and with an average of 200 network realizations. The number of removed nodes decreases at each step. The symbols are the results from simulations and the lines are exponential fits for the results. The slope of the decay λ_K depends on the targeted degree.

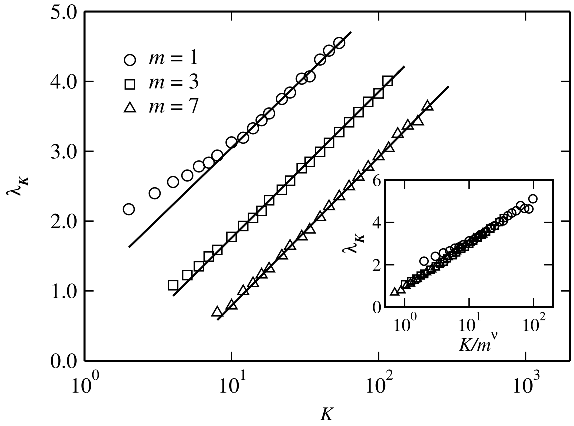

Fig. 4. Dependence of λ_K on the targeted degree K for different values of m. The values of λ_K are obtained from the slope of $\langle s_K \rangle$ versus t (see Fig. 3). We obtain a logarithmic dependence $\lambda_K \sim \delta \ln(K)$, with $\delta = 0.89$. In the inset we show the collapse of the data for different m rescaling K by a factor $m^{-\nu}$, with $\nu = 1.25$.

with $\delta \approx 0.89$, $\nu = 1.25$, and $C_\lambda = 0.41$. Some deviations from this logarithmic growth are observed for small values of K, specially in the case for $m = 1$. The curves for different values of m can be collapsed by the factor m^ν as seen in the inset of Fig. 4.

To demonstrate the consistence of our findings we can verify that the sum of the fractions removed at each step, given by Eq. (3) amounts to the total removed fraction, given by Eq. (2). Approximating the sum by an integral we have $\langle S_K \rangle = \int_1^{T_K} C_K e^{-\lambda_K t} dt = C_K \lambda_K^{-1} K^{-\delta}$, where we have neglected the term of order $e^{-\lambda_K T_K}$. From Eq. (2) we obtain that

$$C_K = \lambda_K e^{\lambda_K} \langle S_K \rangle \sim K^{\delta - \alpha} \lambda_K. \quad (5)$$

This prediction is confirmed by the numerical results of Fig. 5.

Finally we can investigate the behavior of the total duration T_k of the decomposition process. The total duration is the number of steps until no more nodes with the targeted degree are present. A crude estimation of T_K can be made by finding the first step where on average only one node is removed from the network $s_K(T_K) = N^{-1}$. Taking this together with Eq. (3) we obtain

$$T_K = \ln(NC_K)/\lambda_K. \quad (6)$$

In Fig. 6 we compare the average duration obtained by sampling 200 network realizations with the prediction of Eq. (6). One can see that, for large values of K, the numerical result follows closely the expected behavior providing evidence for the consistency of our results.

We have investigated a method of iterative decomposition of BA scale-free networks. Our method consists of successively removing all the nodes with a certain targeted degree. We observe that the fraction of nodes removed at each step decays exponentially with a characteristic parameter controlling the decay λ_K depending on the desired targeted degree. The total

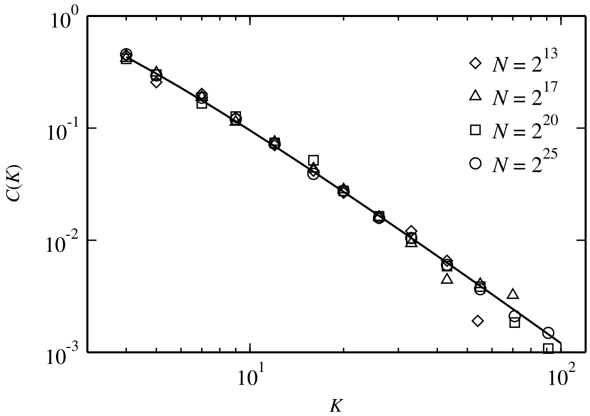

Fig. 5. Dependence of the initial fraction of removed nodes as a function of the targeted degree K. We show data for $m = 3$ and different network sizes. The continuous line is given by Eq. (5).

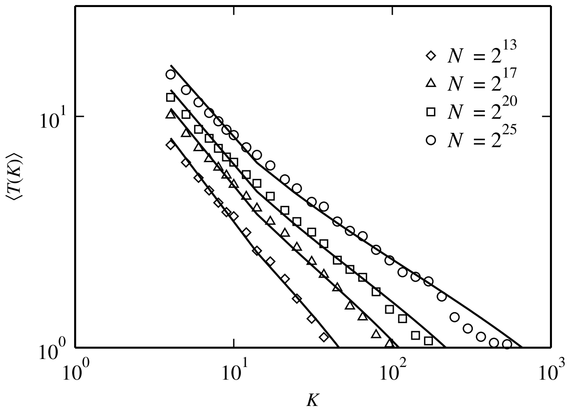

Fig. 6. Average total removal duration as a function of the targeted degree K. The total removal duration is the number of steps until the process stops. The continuous line is given by Eq. (6).

fraction removed at the end of the process decays as a power law with the same exponent as the degree distribution of the original Barabasi–Albert network.

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