A Letters Journal Exploring
the Frontiers of Physics

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EPL, **85** (2009) 20005

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

January 2009

EPL, **85** (2009) 20005 www.epljournal.org
doi: 10.1209/0295-5075/85/20005

On the robustness of q-expectation values and Rényi entropy

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received 12 November 2008; accepted in final form 18 December 2008
published online 28 January 2009

PACS 05.20.-y – Classical statistical mechanics
PACS 02.50.Cw – Probability theory
PACS 05.90.+m – Other topics in statistical physics, thermodynamics, and nonlinear dynamical systems

Abstract – We study the robustness of the functionals of probability distributions such as the Rényi and nonadditive S_q entropies, as well as the q-expectation values under small variations of the distributions. We focus on three important types of distribution functions, namely i) continuous bounded, ii) discrete with finite number of states, and iii) discrete with infinite number of states. The physical concept of robustness is contrasted with the mathematically stronger condition of stability and Lesche-stability for functionals. We explicitly demonstrate that, in the case of continuous distributions, once unbounded distributions and those leading to negative entropy are excluded, both Rényi and nonadditive S_q entropies as well as the q-expectation values are robust. For the discrete finite case, the Rényi and nonadditive S_q entropies and the q-expectation values are robust as well. For the infinite discrete case, where both Rényi entropy and q-expectations are known to violate Lesche-stability and stability, respectively, we show that one can nevertheless state conditions which guarantee physical robustness.

Introduction. – Thermostatistical quantities such as entropy are expressed as functionals of probability distributions. For these quantities to be physically meaningful they should not change drastically if the underlying distribution functions are slightly changed. In practical terms, the unavoidable experimental uncertainty in determining the distribution function should not cause the thermodynamical quantities to fluctuate wildly, or even diverge. It is therefore of elementary interest to clarify and check that thermodynamical quantities are robust under small variations of the distribution functions. This interesting point was first raised by Lesche in 1982 [1]. He proved that the Boltzmann-Gibbs (BG) entropy is, in its discrete form, stable (nowadays often referred to as *Lesche-stable*). Furthermore, he proved that Rényi entropy is not, which in principle makes it inappropriate for thermodynamical purposes (unless one restricts its use to a class of distributions for which the behavior is adequate). Later on, Abe [2] proved that the nonadditive entropy S_q, on which nonextensive statistical mechanics is based, is, like the BG entropy, Lesche-stable. This naturally reinforces its admissibility for thermodynamical purposes.

Before entering into the main purpose of the present paper, let us briefly review the mathematical formulation of Lesche-stability. We define probability distributions on a set of W discrete states, $p=\{p_i\}_{i=1}^W$. Let us denote a variation by $p'_i=p_i+\delta p_i$, the L_1 distance being $\| p-p' \|_1=\sum_{i=1}^W |p_i-p'_i|$. Within the class of L_p distances (or even other types of possible distances), the L_1 distance is adopted because it generically does not depend on W (see [1] for further details). Of course, the verification of this convenient fact within the family L_p by no means proves the uselessness of other possible distances (*e.g.*, Kulback-Leibler-based, or Jensen-Shannon-based) within the same or similar context. Nevertheless, the present Lesche's criterion certainly constitutes a paradigmatic one. Another point which obviously is relevant to physics is whether we desire to apply the stability criterion to *all* probabilitiy distributions, or only to a

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restricted class of them, namely those that have physical relevance, in a sense to be appropriately defined. This particular point will be addressed in detail later on.

A functional $Q[p]$ (*e.g.*, an entropic form) is called *stable* (Lesche-stable) if, for every ϵ, one can find a δ such that for *all* W and for *all* p and p' one has

$$\| p - p' \|_1 < \delta \Rightarrow \frac{|Q[p] - Q[p']|}{Q^{\max}} < \epsilon. \qquad (1)$$

Here $Q^{\max}$ is the maximum of the functional. Lesche could show that, under this strict definition of stability, Rényi entropy is unstable. Indeed, he could find examples for p and p' for which —by taking the $W \to \infty$ limit— Lesche-stability is violated [1]. Taking the $W \to \infty$ limit is essential. If one can show that, in the $W \to \infty$ limit, stability is violated, this implies that for some finite W violation is already emerging, *i.e.* the bound ϵ in eq. (1) gets violated for specific distributions p and p' with $\| p - p' \|_1 < \delta$. Hence, the condition is not true for *all* W, the functional thus is Lesche-unstable. Lesche-stability has been used lately to analyze the stability of various generalized entropies. The use of entropies on that basis was suggested as a new validity criterion [1–4]. It is therefore not surprising that it has occasionally lead to some confusion, see, *e.g.*, [5,6]. The discussion of Lesche-stability has also been extended to other entropies [7,8]. If one does not divide by $Q^{\max}$, eq. (1) becomes the traditional continuity condition for a functional and the stability criterion becomes a notion of uniform continuity.

The requirement that a functional $Q[p]$ should only be considered stable if eq. (1) holds for *all* p and p' and W uniformly, is unnecessarily strict for physical systems. In a physical context it is reasonable to call a functional $Q[p]$ *physically robust* if it is continuous on the domain of *physically admissible* distributions p. In the case of continuous distribution functions, admissible requires that the corresponding BG entropy $(-\int p \ln p)$ is positive[1]. In other words, for physical situations it is sufficient to ensure robustness of functionals, *not* necessarily their stability[2]. Although unbounded physical distributions exist, we are not focusing on them here. For all other physically admissible distribution functions robustness is guaranteed. For finite W any probability distribution is admissible.

The discussion of stability and robustness is not restricted to entropies [2–8], but also to other quantities such as the q-expectation values, which naturally occur in the context of formalisms using generalized entropic functionals [9–12]. The q-expectation values (*i.e.*, standard expectation values with the so-called escort distribution [13], proportional to p^q) naturally appear in differential thermodynamic relations whenever the probability distribution presents power law behavior. This can be illustrated with the q-exponential function, $e_q(x) \equiv [1 + (1-q)x]^{\frac{1}{1-q}}$, which, for $q > 1$, asymptotically decays like a power law. Indeed, whenever one takes derivatives of usual expectation values escort expectation values cannot be avoided, since the exponent q emerges due to $\mathrm{d}e_q(x)/\mathrm{d}x = [e_q(x)]^q$. For instance, normalization of the typical q-exponential distribution $\rho(\epsilon) = e_q(-\alpha - \beta\epsilon)$, where β is the inverse temperature and α a normalization constant, requires $1 = \int_0^\infty \mathrm{d}\epsilon\, e_q(-\alpha - \beta\epsilon)$. A simple calculation shows that the derivative $\mathrm{d}\alpha/\mathrm{d}\beta = -\int \mathrm{d}\epsilon [\rho(\epsilon)]^q \epsilon / \int \mathrm{d}\epsilon [\rho(\epsilon)]^q$, which is exactly the escort expectation of ϵ. Another aspect can be illustrated with unimodal distributions. For example, if one has a q-Gaussian distribution [14], its width can be characterized by the $(variance)^{1/2}$ as long as $q < 5/3$. This is not true anymore if $q \geqslant 5/3$ (*e.g.* for $q = 2$, which corresponds to the celebrated Cauchy-Lorentz distribution) since the variance *diverges*. In all cases, however, we can characterize the width by the inverse of the maximal value of the distribution. It happens that this inverse scales like the $(q - variance)^{1/2}$.

Recently it was shown, using Lesche's two explicit examples for p and p' [1], that q-expectation values are unstable on discrete infinite distributions [15]. The first example corresponds to $0 < q < 1$, the second one to $q > 1$.

Example 1): $0 < q < 1$

$$p_i = \delta_{i\,1}, \quad p'_i = \left(1 - \frac{\delta}{2}\frac{W}{W-1}\right) p_i + \frac{\delta}{2}\frac{1}{W-1}. \qquad (2)$$

Example 2): $q > 1$

$$p_i = \frac{1}{W-1}\,(1 - \delta_{i\,1})\,, \quad p'_i = \left(1 - \frac{\delta}{2}\right) p_i + \frac{\delta}{2}\delta_{i\,1}. \qquad (3)$$

Here $\| p - p' \|_1 = \delta$, for any value of W. Specifically, in [15], instability was shown for the q-expectation of an observable $\mathcal{O} = \{\mathcal{O}_i\}_{i=1}^W$ on the discrete index set $\tilde{I} = \{1, \ldots, W\}$, *i.e.*, the expectation, with $q \neq 1$, of $Q[p] = \sum_i P_i^{(q)} \mathcal{O}_i$, where the escort distribution is given by

$$P_i^{(q)} = \frac{p_i^q}{\sum_{j=1}^{W} p_j^q}. \qquad (4)$$

For both examples $\lim_{W\to\infty} |Q[p] - Q[p']| = |\bar{\mathcal{O}} - \mathcal{O}_1|$, where $\bar{\mathcal{O}} \equiv \lim_{W\to\infty} W^{-1} \sum_i \mathcal{O}_i$, which proves instability when $\mathcal{O}$ and K are chosen such that $|\bar{\mathcal{O}} - \mathcal{O}_1| > K > 0$ [15]. This implies that q-expectations are not *uniformly* continuous functionals in the $\lim W \to \infty$. It was concluded in [15] that the instability of the q-expectation value is the general situation, thus suggesting to re-think the use of q-expectation values in nonextensive statistical mechanics. While the result in [15]

[1]In the continuum, entropy functionals such as the BG, Rényi and others are well known to become negative for distributions which include too narrow peaks, a situation which typically corresponds to the low-temperature limit, where the quantum nature of physical systems *must* be taken into account.

[2]If a functional is stable it is always safe to use. Inversely, instability points at the fact that the domain of safe usage is limited. Here, robustness is never used in the sense of trajectories or attractors, as done in other contexts.

is correct in the strict sense of stability used there, this does *not* imply that q-expectation values are not robust either on finite sets —as will be shown here— or for continuous variables with the mentioned physical admissibility and boundedness conditions [16]. Therefore, the final conclusion drawn in [15] that the q-expectation is *in general* unstable under small variations of the probability distributions does *not* hold for physically relevant cases such as continuum distributions and discrete distributions on finite support. Even for a discrete infinite support, robustness is verified, as it will be shown, for paradigmatic physical distributions. Robustness in the above sense is sufficient for virtually all practical physical purposes. In other words the requirements of boundedness and positive entropy exclude the pathological cases of singular distributions and singular variations. The examples used in [15] are representatives of such pathological cases.

In this contribution we primarily discuss stability and robustness of q-expectation values and Rényi entropy for three types of support for distribution functions, the continuous, discrete finite, and discrete infinite. For the continuous case, under the requirement of physically admissible and bounded probability distributions and variations, we show the robustness of q-expectation values and Rényi entropy. We then show that the theorems given in [16] allow to prove robustness for *all* finite discrete sets. Even though it is not possible to immediately use the theorems to make statements about infinite discrete sets, which have been shown to be unstable for Rényi entropy and the q-expectation value (for $q \neq 1$) [1,15], we show how the theorems can be used to derive restrictions so that robustness can be accomplished there as well. We finally discuss the situation for S_q entropy for continuous and discrete finite probability distributions. A discussion on the distinction of discrete finite and infinite cases has been presented on numerical grounds in [17]. It is known that the nonadditive entropy S_q on discrete infinite distributions is robust because it is Lesche-stable [2].

Stability criteria for the q-expectation value for admissible continuous distribution functions. – To make the paper self-contained we first review the stability criteria for the continuum case as discussed in two theorems in [16]. These two theorems determine the robustness criteria for q-expectation values in the continuum. These theorems will be used below to show that not only the two examples of Lesche used in [15] are robust on finite sets, but that this is the case for *all* distribution functions on finite sets.

For notation, in the continuum, the escort distribution reads $P^{(q)}(x) \equiv \frac{\rho(x)^q}{\int \mathrm{d}x' \rho(x')^q}$, where ρ denotes a continuous probability distribution. The expectation value of an observable $\mathcal{O}(x)$ under this measure is $\tilde{Q}[\rho] = \int \mathrm{d}x P^{(q)}(x)\mathcal{O}(x)$, and its total variation reads $\delta\tilde{Q}[\rho] = \tilde{Q}[\rho+\delta\rho] - \tilde{Q}[\rho]$. Here we use $\tilde{Q}[\rho]$ to distinguish from the discrete case.

The case $0<q<1$. The following theorem proves that, for $0<q<1$, instability only can happen for singular distributions ρ. In the theorem $\parallel \mathcal{O} \parallel_\infty = \sup\{|\mathcal{O}(x)| : x \in [0,1]\}$ denotes the so-called supremum or uniform norm, which is just the smallest upper bound of $|\mathcal{O}|$.

Theorem 1. *Let* $0<q<1$. *Let* $0<\rho$ *be a non-singular probability distribution on* $I=[0,1]$. *Let* $G=\int_I \mathrm{d}x\, \rho(x)^q$ *and let* $0<\tilde{\delta}^q = \mu G/4$, *for* $0<\mu<1$, *and* $\delta\rho$ *be a variation of the distribution such that,* $\int_I \mathrm{d}x|\delta\rho| = \delta \leqslant \tilde{\delta}$, *and* $0<\rho+\delta\rho$ *is positive on* I. *Furthermore, let* $0<\mathcal{O}$ *be a strictly positive bounded observable on* I, *then there exists a constant* $0<c<\infty$, *such that*

$$|\tilde{Q}[\rho] - \tilde{Q}[\rho+\delta\rho]| < c\delta^q. \tag{5}$$

Moreover, $c = 4G^{-2} \parallel \mathcal{O} \parallel_\infty (1 + \parallel \mathcal{O} \parallel_\infty \parallel \mathcal{O}^{-1} \parallel_\infty)/(1-\mu)$.

The theorem states that, for positive bounded observables, q-expectation values are robust whenever the distribution ρ is non-singular[3]. The class of singular distributions is therefore the only class of distributions that contain all possible violations to stability for $0<q<1$, as long as the observable $\mathcal{O}$ is bounded on domain I. The corresponding example in [15] explicitly converges toward a singular distribution in the $W \to \infty$ continuum limit and thus violates stability.

The case $q>1$. In contrast to the $0<q<1$ case, instability for $q>1$ is not primarily due to singular distributions ρ, but due to the variation $\delta\rho$ having singular parts, *i.e.*, due to an unbounded $\delta\rho$. Note, that for bounded $\delta\rho$ to exist, ρ also has to be non-singular. To keep $\int \mathrm{d}x[\rho(x)]^q$ and $\int \mathrm{d}x[\rho(x)]^q\mathcal{O}$ finite, we further restrict ρ to be bounded.

Theorem 2. *Let* $q>1$ *and let* $m>0$ *be an arbitrary but fixed constant. Let* $0<\rho$ *be a probability distribution on* $I=[0,1]$. *Let* $\delta\rho$ *be variations of* ρ, *i.e.* $\rho+\delta\rho>0$. *Let* $B>0$ *be an arbitrary but fixed constant. Let the variations* $\delta\rho$ *be uniformly bounded in the m-norm, i.e.* $\parallel \delta\rho \parallel_m < B$, *by this constant* B. *Further, let* $\parallel \delta\rho \parallel_1 = \delta$ *and let* $0<\mathcal{O}$ *be a strictly positive bounded observable on* I. *Let* $\tilde{\delta}$ *be an upper bound for the size of the variations* δ *such that,* $(2^{1/q}-1)^{q/\gamma}(B^{q-\gamma} \parallel \mathcal{O} \parallel_\infty \parallel \mathcal{O}^{-1} \parallel_\infty)^{-1/\gamma} > \tilde{\delta} > 0$, *where* $\gamma = (m-q)/(m-1)$, *then there exists a constant* $0<R<\infty$, *such that*

$$|\tilde{Q}[\rho] - \tilde{Q}[\rho+\delta\rho]| < R\delta^{\gamma/q}, \tag{6}$$

and R *does not depend on the choice of* ρ.

Theorem 2 states that, for positive bounded observables, q-expectation values are robust whenever the distributions ρ are uniformly bounded[4]. Excluding unbounded variations from consideration therefore is

[3]When all considered ρ are bounded by the same bound $0<\rho<B$, then the constant c does not depend on the choice of ρ and $\tilde{Q}$ is absolutely continuous on this domain.

[4]When all considered distributions are bounded by the same upper bound ($m \to \infty$ and $\gamma \to 1$), then R can be chosen independent of ρ and $\tilde{Q}$ becomes uniformly continuous.

sufficient to guarantee stability for the $q > 1$ case in a general setting. In the corresponding example in [15], when formulated in the continuum limit, stability is violated by using unbounded variations[5].

Both theorems in [16] are analytical statements about the continuity properties of the q-expectation value as a non-linear functional without any reference to thermodynamics. They provide a useful and flexible mathematical tool to analyze continuity properties for discrete sets as well as for the continuous case on $[0, \infty]$.

Stability criteria for the q-expectation value for discrete finite probability functions. – We can now use the above theorems to prove the robustness of q-expectations for bounded observables on finite discrete sets $i \in I_W \equiv \{1, 2, \ldots, W\}$. For this we map the discrete probability distribution onto the continuous interval $x \in [0, 1]$ by identifying probabilities $\{p_i\}_{i=1}^W$ with probability densities $\rho(x) = W p_i$ for $x \in [(i-1)/W, i/W]$, *i.e.* step functions on $[0, 1]$. The observable $\{\mathcal{O}_i\}_{i=1}^W$ gets identified with the step function $f(x) = \mathcal{O}_i$ for the associated interval $x \in [(i-1)/W, i/W]$. Clearly, $0 \leqslant \rho \leqslant W$ for all possible distributions of this kind and so are all possible variations since $|\rho(x) - \rho'(x)| < W$. Since the observable $f(x)$ is bounded, all conditions needed for Theorems 1 and 2 are met. Let $\gamma = q$ for $0 < q < 1$ and $\gamma = 1/q$ for $q > 1$. For some given constant $0 < \tilde{\delta}$ there exists a constant C, such that for all $|\rho - \rho'| < \delta < \tilde{\delta}$ it follows that $C\delta^\gamma > |\tilde{Q}[\rho] - \tilde{Q}[\rho']|$ where $\tilde{Q}[\rho] = \int_0^1 \mathrm{d}x\, [\rho(x)]^q f(x) / \int_0^1 \mathrm{d}x\, [\rho(x)]^q$ is the q-expectation for the continuous case. Now $\tilde{Q}[\rho] = \sum_{i=1}^W W^{-1} [W p_i]^q \, \mathcal{O}_i \,/ \sum_{i=1}^W W^{-1} [W p_i]^q = \sum_{i=1}^W p_i \, \mathcal{O}_i \,/ \sum_{i=1}^W p_i^q = Q[p]$, where $Q[p]$ is the q-expectation on I_W. Moreover, $\| \rho \|_1 = \int_0^1 \mathrm{d}x\, |\rho(x)| = \sum_{i=1}^W |p_i| = \| p \|_1$ and consequently one can pull back the result to the discrete case, which completes the proof.

Comments on discrete infinite probability functions. – For discrete infinite distribution functions it was shown [15] that the q-expectation value is unstable. However, it is possible to state conditions under which robustness can be ensured. This can be done, for instance, in the following way.

Discrete finite sets are closely related to continuous compact sets in the sense that discrete sequences can be mapped into the compact interval with step functions, as discussed above. In the same spirit, discrete infinite sets are intimately related to the continuous unbounded set $[0, \infty]$, since again one can map the discrete infinite sequence into the continuous case in terms of step functions. If one can find conditions which define classes of distribution functions on $[0, \infty]$ that guarantee continuity or absolute continuity of the functional, *i.e.*, the q-expectation, the same conditions are sufficient for probabilities on infinite discrete sets. Such classes can simply be derived by using suitable differentiable monotonous functions, $g : [0, \infty] \to [0, 1]$. Let g' denote the derivative of g and g^{-1} the inverse function of g. This maps the distribution function ρ, defined on $[0, \infty]$, to a distribution function $\tilde{\rho}(y) = \rho(g^{-1}(y)) g'(g^{-1}(y))^{-1}$ on $[0, 1]$ and also the observable $\mathcal{O}$ on $[0, \infty]$ gets mapped to $\tilde{\mathcal{O}}(y) = \mathcal{O}(g^{-1}(y)) g'(g^{-1}(y))^{q-1}$. Now one can apply the conditions used for the Theorems 1 and 2 on $[0, 1]$ and pull them back to $[0, \infty]$. It should be noted that different maps g lead to different, yet consistent boundedness conditions and decay properties for observables $\mathcal{O}$ and distributions ρ on $[0, 1]$.

To give an explicit example, let us consider the following problem. Suppose we consider $\bar{q}$ -exponential distributions of the form $\rho(x) \propto e_{\bar{q}}(-\beta x) \equiv [1 - (1 - \bar{q})\beta x]^{\frac{1}{1-\bar{q}}}$ for $\bar{q} \geqslant 1$ and some $\beta > \beta_0$, and we want its first N moments under the q-expectation,

$$\langle x^m \rangle_q \equiv \frac{\displaystyle\int \mathrm{d}x [\rho(x)]^q x^m}{\displaystyle\int \mathrm{d}x [\rho(x)]^q}, \tag{7}$$

to be robust with respect to g, $(m \leqslant N)$. For a more detailed discussion on moments under q-expectations, see [18]. Assuming $q > 1$, let us take $g(x) = 1 - 1/(1 + x)^\gamma$; then $g'(x) = \gamma(1 + x)^{-\gamma - 1}$. The boundedness condition for the observables immediately requires $\gamma > N/(q - 1) - 1$ and the decay property for the distributions implies $\bar{q} < 1 + 1/(\gamma + 1)$. For $\bar{q} = 1$ and $m = 1$ the example is the q-expectation value of the energy of the quantum harmonic oscillator,

$$\langle E \rangle_q = \frac{\sum\limits_{n=1}^{\infty} n [e_1^{-\beta n}]^q}{\sum\limits_{n=1}^{\infty} [e_1^{-\beta n}]^q} = \frac{1}{4} \sinh^{-2} \left(\frac{\beta q}{2} \right), \tag{8}$$

which shows continuity in β and robustness of the q-expectation under variations of the exponential distribution[6].

Robustness of Rényi entropy for continuous and discrete finite distribution functions. – Here it suffices to discuss the case $0 < q < 1$ since for $q > 1$ Rényi entropy

$$S_q^{\mathrm{R}} = \frac{\ln \sum\limits_{i=1}^{W} p_i^q}{1 - q} \tag{9}$$

[5] The proof was carried out on the unit interval $I \in [0, 1]$. This does not present a loss of generality, since the proofs can be extended to any bounded interval. For unbounded intervals, the proof gets more involved and requires to fix conditions that relate boundedness conditions of the observable and the decay properties of ρ.

[6] In what concerns the use of continuous distributions for calculating entropies and similar quantities, such as eq. (7), the reader must be aware of a relatively well-known difficulty. If we make a change of variables $y = f(x)$, say in eq. (7), we immediately see that the result is not invariant. In the spirit of Kullback and Leibler entropy [19] the problem is easily resolved in terms of a *reference distribution* $r(x)$, which, except at infinity, nowhere vanishes. For example, for $q > 0$, the quantity in eq. (7) would be replaced by $\frac{\int \mathrm{d}x r(x) [\rho(x)/r(x)]^q x^m}{\int \mathrm{d}x r(x) [\rho(x)/r(x)]^q}$.

Table 1: Table of stability/Lesche-stability and robustness for the functionals S_q, S_q^{R} and the q-expectation value for the underlying nature of the distribution function, *i.e.* continuous, discrete finite or discrete infinite. In the continuum, admissible means that a non-negative entropy is required. Boundedness automatically guarantees robustness [16]. The term *Lesche-stable* is used for infinite distribution functions when eq. (1) holds, *stable* is used when there is no division by a maximum taken in eq. (1). *Robustness* is used for bounded distributions in the continuum and for discrete finite probability functions. For the q-expectations for the discrete infinite case, robustness is understood under the decay properties of the distributions, as specified in the text. Lesche-stability and stability are sufficient but not necessary for robustness. S_{BG} and standard expectation values ($q=1$) are Lesche-stable and stable, respectively.

	S_q, for $q>0$	S_q^{R}, for $q>0$	q-expectation value, for $q>0$
bounded continuous (physically admissible)	robust	robust	robust [16]
discrete finite (W finite)	robust	robust	robust
discrete infinite ($\lim W \to \infty$)	Lesche-stable [2]	Lesche-unstable [1]; robust for typical physical cases	unstable [15]; robust for typical physical cases

strictly speaking ceases to be a proper entropy because it is not concave. Substituting the probabilities p_i by step functions $\rho^{(W)}(x)=Wp_i \leqslant W$ for $x\in[(i-1)/W, i/W]$ which represent the discrete probability p_i as a probability density on $[0,1]$, we get

$$S_q^{\mathrm{R}} = \ln W + \frac{\ln \int_0^1 \mathrm{d}x\,[\rho^{(W)}(x)]^q}{1-q}. \tag{10}$$

Boundedness of the distributions allows to use propositions (4), (6) and (8) in [16] which have been used to prove Theorem 1. Note that, due to the upper bound W, it follows that $\int \mathrm{d}x\rho(x)\geqslant W^{q-1}$. Let $\tilde{\delta}=\mu W^{q-1}/4$ and $\|\,\delta\rho\,\|_1=\delta<\tilde{\delta}$, as in [16]. Now we get

$$|S_q^{\mathrm{R}}[\rho]-S_q^{\mathrm{R}}[\rho+\delta\rho]| = \left|\ln \frac{\int_0^1 \mathrm{d}x\,[\rho(x)]^q}{\int_0^1 \mathrm{d}x\,[\rho(x)+\delta\rho(x)]^q}\right|. \tag{11}$$

Using propositions (4) and (6) from [16] one finds $1-a\delta^q<|\int_0^1 \mathrm{d}x\,\rho(x)^q/\int_0^1 \mathrm{d}x\,(\rho(x)+\delta\rho(x))^q|<1+a\delta^q$ and $a=4W^{1-q}/(1-\mu)$. Since $|\ln(1+x)|<2|x|$, for $|x|\ll 1$ it follows that for sufficiently small δ,

$$|S_q^{\mathrm{R}}[\rho]-S_q^{\mathrm{R}}[\rho+\delta\rho]|<2a\delta^q. \tag{12}$$

This shows both the uniform continuity of the continuous Rényi entropy for the class of uniformly bounded probability distributions in $L_1([0,1])$, and the absolute continuity of Rényi entropy for probabilities on finite sets. Similar arguments show robustness also for $q>1$, despite lack of concavity of S_q^{R}.

Robustness of the entropy S_q for continuous and discrete finite distribution functions. – One can prove robustness of the nonadditive entropy

$$S_q = \frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \tag{13}$$

for discrete finite sets by mapping the probabilities $\{p_i\}_{i=1}^W$ onto a distribution on ρ on $[0,1]$ by step functions, as above. Since all step functions ρ that represent some $\{p_i\}_{i=1}^W$ are bounded by W, it is sufficient to prove robustness for the continuous case. ρ is from a uniformly bounded class of distribution functions on $[0,1]$, *i.e.*, there is a constant $W>0$ such that all ρ are bounded by W. Consider variations $\delta\rho=\rho'-\rho$ such that both ρ and ρ' are bounded by W and $\|\,\delta\rho\,\|_1=\delta$, where δ is sufficiently small. Now, $|S_q[\rho]-S_q[\rho+\delta\rho]|=|q-1|^{-1}|\int_0^1 \mathrm{d}x\,\rho^q-\int_0^1 \mathrm{d}x\,(\rho+\delta\rho)^q|$. It therefore is an immediate consequence of propositions (6) and (4) in [16], that for $0<q<1$, $|S_q[\rho]-S_q[\rho+\delta\rho]|<4\delta^q$. For $q>1$ we use propositions (11) and (13) to find $|S_q[\rho]-S_q[\rho+\delta\rho]|<R\delta^{\frac{1}{q}}$.

In what concerns discrete infinite distribution functions, S_q has been shown to be Lesche-stable [2].

Conclusion. – To summarize, we discussed the concept of physical robustness in contrast to the more restrictive mathematical stability of thermostatistical functionals under variations of their underlying distribution functions. We argue that while several important functionals, such as the Rényi entropy or the q-expectation value are unstable in the strict sense, restriction to physically relevant distribution functions, ensures robustness of these functionals. For a distribution to be physically relevant we require that its associated entropy be non-negative. We further restrict to distributions which are bounded in the continuum. This excludes, for example, distributions involving Dirac deltas. Our results are summarized in table 1, where we indicate the type of stability, Lesche-stability or robustness for the functionals S_q, Rényi entropy S_q^{R} and the q-expectation value, for the paradigmatic types of distribution functions —continuum, discrete finite, and discrete infinite— that we have focused on here. The term Lesche-stable is used when eq. (1) holds, stable refers to the situation where no division by a maximum is taken in eq. (1), and robustness —in the above-defined sense— is found for admissible (non-negative entropy) and bounded distributions in the continuum and for discrete finite probability functions. In the case for discrete infinite distribution functions which are known to cause instabilities for some functionals [1,15], one can show that there exist paradigmatic robust examples once decay properties of the distributions are specified. In this context one can show that systems such as the harmonic oscillator are robust under variations of the (inverse) temperature. One can of course think of physical distribution functions which have positive BG entropy but are unbounded, such as, *e.g.*, power law divergences. These cases remain to be discussed but exceed the present scope.

We conclude by stating that the concept of stability might be overly strict for physical applications. This is in accordance to conclusions drawn in [20]. When limited to the class of physically admissible and bounded distribution functions, it is conceivable that physical robustness of virtually all thermodynamic functionals will be guaranteed.

* * *

We acknowledge fruitful remarks by E. M. F. Curado and C. Beck. RH and ST acknowledge great hospitality at CBPF. This work was partially supported by Faperj and CNPq (Brazilian agencies), and Austrian Science Fund FWF Project P19132.

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