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Introduction

Nonlinear dynamics in meso and nano scales: fundamental aspects and applications

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This introduction to the special issue, *Nonlinear dynamics in meso and nano scales: fundamental aspects and applications*, gives a short overview about different contexts and current challenges posed by the emergence of nonlinearities at meso and nano characteristic sizes. It also addresses different aspects related to classical and quantum chaos. Moreover, it comments on the articles in this thematic publication, briefly summarizing their relevance in helping to understand the uprise of chaos and complex behaviour at those small scales.

Keywords: nonlinearity; nanoscopic systems; mesoscopic systems; classical chaos; quantum chaos

1. Introduction

Once the very *fundamental* laws of nature are conceivably exactly the same for any process, why do we divide physics into great areas such as cosmology, statistical mechanics, condensed matter, atomic and molecular physics, high-energy physics, etc.? The answer, to a large extent, relies on the type of methodology, concepts and principles adopted to explain and comprehend a particular phenomenon. In fact, the specific framework considered to treat a system is closely related to its resulting collective interactions and the amount and degree of organization of matter, which are the aspects setting the characteristic scales of energy, size and time. For instance, generally we do not start with the strong force (responsible for maintaining the atomic nucleus together) to discuss the chemical properties of the atomic elements. We rather consider the nuclei as 'given' structures of positive charges and then analyse their interaction with the electrons through the electromagnetic force. This is so because of the disparate

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energy and spatio-temporal scales associated with each level of description, e.g. while the spatial scales of nuclei are in the interval 10^{-15}–10^{-14} m, that of the atom is of the order of 10^{-10} m. Moreover, such a hierarchical depiction procedure is applied not only when the physical mechanisms involved are distinct at the various scales. The same interaction can be handled differently according to the way we group the elementary parts of a system. For instance, although in both cases we are dealing with gravity and general relativity, we can use different approaches to study the dynamics of galaxies [1] or of the (matter) constituents of the universe as a whole, where the latter can even be treated as a gas [2].

Despite the above comments, the huge diversity of natural phenomena hampers one in maintaining that a particular behaviour is restricted only to a specific scale. Let us take the very relevant case of nonlinearity and to consider the domain of phenomena relative to our quotidian lives, of which a lay person has an 'awareness' about. It typically comprises 10^{-4}–10^{7} m spatial scales, say between the extremes of naked-eye acuity and intercontinental travel distances. In such everyday context, well described by classical physics, nonlinear dynamics is commonplace. Hence, the practical necessity of understanding and predicting nonlinearity at large scales, such as oscillations in grandstands, ecological tradeoffs in tropical rain forests and the climate, is of no surprise. All these problems, displaying nonlinear interactions, give rise to non-trivial intricate evolutions, sometimes leading to qualitatively similar patterns like chaotic structures.

Historically, the study of the impact of nonlinearities on (large) macroscopic systems starts with the pioneering work of H. Poincaré at the end of the nineteenth century. He analysed the three-body problem, having, as initial motivation, to verify the Solar System stability. Indeed, the intrinsic mathematical structure of Newton's second law (and also, in a broader context, of Einstein's relativity) allows nonlinear dynamics through the nature of the interactions. This can result in rich features such as: multi-stability, sensitivity to initial conditions, mixing properties and topological structures in the phase space, proliferation of periodic orbits, etc. [3]. Nevertheless, nonlinear behaviour is not restricted to mechanics or, more generally, to dynamical systems. It is associated with any process that can be described by nonlinear equations [4]. Thus, in our surroundings, from the millimetric elastic–plastic vibrations to the thousand kilometre wide weather systems, nonlinearity is everywhere. So, a natural question is: for which size scales can we find such complex behaviour?

Nowadays, we know that at the opposite extreme, in the very small scales of 10^{-15} m, nonlinearity is essential to describe nuclear physics and quantum chromodynamics [5,6]. But what happens when we consider nano (10^{-9} m) and meso (10^{-6} m) scales, i.e. systems having from a few tens to a few thousands of atoms along a characteristic direction? As we will see throughout this introductory article, at these scales, nonlinear effects may also play a fundamental role in determining the system behaviour. They take place either owing to the signature of chaos in the quantum mechanics (QM) domain (quantum chaos), or owing to the involved nonlinear forces and dynamical equations describing the problem, if the processes are of classical nature. Here, we address some fundamental aspects in this respect. We discuss whether nonlinear classical physics can rule the dynamics at such small scales, as well as in what sense non-relativistic QM, whose Schrödinger equation is essentially linear, can lead to 'chaos'. These and other relevant issues are considered in §2.

Finally, we should mention that besides the pure scientific interest in the above questions, there is also a direct or indirect motivation in technological applications. Since the dare by Richard Feynman in the 1950s, offering a prize to the first person to build an electrical motor smaller than 1/64 cubic inch, a whole realm of micro and nano devices arose. Illustrations are microsensors, microelectromechanical devices (MEMS), and more recently, nanoelectromechanical systems (NEMS), with sizes far below the Feynman provoking threshold. The advantages of these systems are many-fold, and indeed significant, owing to the enormously improved measurement sensitivity, much lower response times and very small energetic dissipation (thus lower energy consumption). However, nowadays this progressive miniaturization of devices is attaining the quantum limit, just above the bounds of the uncertainty principle, so that: gating performance becomes affected by confinement hindrances owing to tunnelling effects, miniaturized transistor arrays become heterogeneous owing to the inhomogeneous proportion of impurities, or even relativistic effects start to be an issue, like in graphene Dirac cones or massless Dirac fermions. These are examples of the new obstacles that need to be overcome. Interestingly enough, it has been proposed that in many instances, nonlinear effects could help to improve device characteristics [7,8], even in quantum computing [9]. Therefore, there is a practical necessity to undergo the numerous queries posed and further develop the understanding of nonlinearities at such small scales.

A comment: the special issue, *Nonlinear dynamics in meso and nano scales: fundamental aspects and applications*—to which this piece is an introduction—addresses both fundamental and practical challenges in the area (see §3). By means of different perspectives, approaches and case studies, it discusses how nonlinear features and their consequences emerge in systems at such a size range. It offers a general overview on how either classical or quantum physics can describe their main characteristics and properties.

2. Nonlinearities at nano and meso scales

There are few primordial open problems in physics, challenging our more basic understanding of nature. Some examples (but not a complete list) are: cosmic inflation actual mechanisms, dark matter and energy, baryon asymmetry (matter anti-matter unbalance), mass generation, quantum gravity, high-critical-temperature superconductors, the exact path to turbulence, and fluid-dynamics stability in three dimensions. Among them, we can also include the exact border between classical and QMs and their transition [10]. This latter is closely related to our present discussion on nonlinear dynamics in nano and meso 'worlds'. Indeed, what can lead to nonlinear behaviour at 10^{-9}–10^{-6} m characteristic sizes? Could typical classical dynamics rule at those scales? Would pure QM, described by a linear Schrödinger equation, give rise to chaos? Is an intrinsic nonlinear QM possible? And finally, can the quantum–classical border itself be responsible for nonlinearities?

Surely, all these are important questions and, in many aspects, far from having definitive answers, each deserving a whole special issue dedicated to it. Here, we just outline some facts and results, putting the above queries in a broader perspective, and presenting partial answers to them.

(*a*) *How small can a classical nonlinear phenomenon be?*

To describe nano and meso scale systems, the very first point is to determine whether the dynamics of interest can be described by classical physics or if it already behaves quantum mechanically [11,12]. Actually, this may not be a simple task, although some experimental [13] and theoretical [14] frameworks have already been proposed to settle the issue.

An essential ingredient is the parameter range associated with the phenomenon to be analysed. For a given system, specific behaviours are associated with different underlying physics related to its parameter ranges. For instance, consider the electrical–mechanical properties of NEMs and MEMs. One may be interested in the vibrational states of the whole structure, meaning spatial dimensions of a few nanometres to micrometres. Alternatively, the focus may be on electronic states and charge transport in a confining region of the device (e.g. an embedded quantum dot). Then, the actual values of characteristic masses, energies, temperature and relaxation times will define what physics context the sought phenomenon belongs to [15]. In the case of vibrational states, if temperatures are not low enough to reduce the thermally induced Brownian motion to the same energetic order of the lowest quantized phonon modes [16], then the vibrations can be treated as classical mechanical waves. In such a case, a description in terms of nonlinear classical resonators can be used, with the dynamics displaying strong chaotic behaviour [17,18]. Moreover, for electronic states, if impurities, intrinsic disorder, imperfect border walls or incoherence are present, typical quantum correlation lengths are much smaller than the system size. Hence, many of the quantum characteristics of electrons are lost [19–21], and the problem in some respects can be faced as that of classical carriers [22].

We should also note that, conceivably, a large system may have as the origin of its particular behaviour, nonlinearities generated at small scales (meso and even nano [23]), which can then, owing to some amplifying mechanism (e.g. fluctuation–dissipation thermal effects [24]), scale up to larger scales. This direct connection between nonlinearities at different spatial sizes—where for the larger ones, the physics involved is certainly classical—may render the microscopic nonlinearities to be classical in nature as well (a nice example being chaos in chemical reactions [25]). Such a hierarchical process, when being the case, can be formalized in a more rigorous way, e.g. by means of kinetic theory and the Boltzmann equation [26].

Summarizing, very small spatial sizes are just one of the factors determining which is the fundamental physics governing a system. For a phenomenon to be classical, the common wisdom is for $\hbar$ to be small compared to relevant parameters of the system. Size is one of these parameters, but is not the unique one. It may turn out that processes taking place along hundreds of angstroms (just 100 atoms in a row) are already sufficiently large to obey laws that—although having a quantum ingredient—in principle are valid only macroscopically, such as black-body radiation [27].

(*b*) *Can quantum mechanics be intrinsically nonlinear?*

Assume that a process is, in fact, quantum at the characteristic sizes we shall study. We can ask then if *exactly* the same type of nonlinear effects emerging in classical physics at those scales (cf. §2*a*) could occur in the quantum case. The above qualification 'exactly' is not just rhetorical. Here, it concretely means the kind of effects and behaviour resulting from a classical system that is described by nonlinear dynamical equations. In other words, for non-relativistic microscopic processes down to the atomic level, i.e. at scales of order 10^{-10} m (for smaller scales, see [28]), could the Schrödinger equation in particular and the QM formalism in general have a nonlinear mathematical structure?

Very fundamental debates about the subject have been waged. Given the huge success of the present linear quantum theory in predicting all known microscopic phenomena, many scientists argue that nonlinearity should not be part of QM. Of course, such objections are flawed if nonlinear effects are too small, and very precise measurements would be necessary to probe them (outside our current experimental technology). In the same direction, it could happen that the type of phenomena where the nonlinear aspects of QM are really important have not been discovered yet (although this is unlikely given the near 100 years of QM)!

It is true that some proposals for a nonlinear quantum theory have been put forward (for reviews, see [29,30]), especially those by Weinberg [31,32] as well as an older and interesting idea by Bialynicki-Birula & Mycielski [33]. Importantly enough, such constructions provide a framework to test possible nonlinear effects in QM. Experimental tests, however, have not shown any nonlinear behaviour as discussed, e.g. in [30,34]. Also, nonlinearities could cause several theoretical difficulties [30,34,35]. Among them, one is particularly interesting because of its very abstract algorithmic character. It has been shown that a nonlinear QM would lead [34] to a polynomial-time solution (P) for non-deterministic polynomial-time (NP) problems that up to the present time, only have non-polynomial-time algorithmic solutions known. However, there are many theoretical reasons to believe that indeed $\mathrm{P} \neq \mathrm{NP}$ [36] (although a rigorous proof is still waiting for a US$ 1,000,000 prize (http://www.claymath.org/millennium/)). So, it seems that even pure mathematics is 'against' intrinsic nonlinearities in QM.

Therefore, presently we should not expect intrinsic nonlinear features directly from QM dynamics via a fundamental (first principles) nonlinear Schrödinger equation. Nevertheless, we should mention that complex quantum mechanical systems, where either many constituents are strongly interacting or the environment has an important influence on the system [37–39], may result in an effective time evolution described by a nonlinear dynamical equation (e.g. as in the Gross–Pitaevskii equation in a mean-field treatment of many bosons), derived from linear theory [40–42]. In such a case, many of the interesting manifestations of nonlinearity in the classical case may also be present in truly QM problems [43–45].

(*c*) *How can linear quantum mechanics lead to nonlinear behaviour?*

Considering the 'traditional' QM theory, i.e. the usual (linear) Schrödinger equation, the question is if we can have typical nonlinear behaviour—in the classical-physics sense—out of a fully linear quantum dynamics. Although what we call quantum chaos nowadays is already nearly 40 years old (see the discussion below), to address exactly the point raised above, we comment on a more recent constructive debate, appearing about 15 years ago.

In 1994, a very original system was proposed [46,47]. The idea was to consider spin-1/2 particles propagating through a chain of spaced magnetic regions Σ_n ($n=0,1,\ldots$), each single region Σ_n being composed of a linear array of two types of magnets, A and B, aligned normal to the chain axis and forming a fixed angle with respect to each other. The successive arrays, furthermore, should follow the Fibonacci pattern $\Sigma_0=$ A, $\Sigma_1=$ B, $\Sigma_2=$ BA, $\Sigma_3=$ BAB, $\Sigma_4=$ BABBA, $\Sigma_5=$ BABBBABAB, and so on. The interesting dynamical variable here is the spin rotation angle β_n (with respect to a reference axis) after crossing each magnetic region Σ_n. It was shown that the rotation angles were exponentially sensitive to the exact values of the magnetic field strengths. In other words, it was a purely quantum system showing exponential sensitivity, in principle, characteristic of nonlinear classical problems. The whole discussion was raised [48,49] owing to the fact that, in the classical parlance, exponential sensitivity owing to nonlinearities is related to the initial conditions. However, in the mentioned example, the sensitivity is associated with the system parameters and not with the initial preparation of the system, namely the starting wave function. So, strictly speaking, this system is not chaotic in the classical sense. Presently, a fully closed quantum system displaying any exponential sensitivity in its initial conditions remains unknown (however, for a classical–quantum interaction, see [50]).

The above example illustrates that, since the concepts of trajectories and phase space in classical mechanics must be faced differently in QM [51,52], two fundamental features of nonlinear systems: (i) exponential sensitivity to the initial conditions and (ii) complex topological structures of the phase space (strange attractors, density of periodic orbits, mixing filamentation, etc.) are absent in the usual QM. Thus, the natural question is whether there is such a thing as quantum chaos. The answer is a bold yes if we rephrase the question as follows: when a classical system presents nonlinear behaviour, does its quantized version show specific distinct characteristics? Michael Berry coined a term for such a type of study, calling it 'quantum chaology' [53,54], for a distinction from chaos in its more traditional classical sense. However, nowadays 'quantum chaos' is used indiscriminately as having Berry's meaning.

What then are the features of a quantum system whose classical analogue is chaotic? When can one say that a quantum system displays chaos? The answer is not unique, once there are many fingerprints indicating quantum chaoticity. It is outside our present scope to review all the properties and historical developments characterizing quantum chaos. Next, we just give a very brief (and incomplete) account on the subject.

Since the fundamental work by E. P. Wigner, F. J. Dyson and M. L. Metha (among others), the random matrix theory (RMT) [55] has been used in different instance, especially, to analyse processes in nuclear physics, a context where the particular forces and their exact actions are still not clearly understood. The idea is quite simple in concept. Suppose that the Hamiltonian H of a system must obey very general properties, e.g. regarding Hermiticity, time-reversal invariance symmetries, spin values, etc. Apart from verifying such conditions, the specific matrix elements of H can be arbitrary and drawn from a certain distribution, e.g. Gaussian. One can consider an ensemble of such matrices and then calculate average properties of the resulting eigenvalues. It turns out that for nuclear problems, such average properties can fit experimental data reasonably well. In a thoughtful insight, Bohigas *et al.* [56] conjectured that the quantum spectra of all chaotic systems should behave as the predictions of RMT, in contrast to regular systems [57]. Many theoretical and experimental works [51,52] along the years have confirmed the hypothesis. Hence, the spectral statistical features described by RMT are nowadays taken as one of the most important signatures of quantum chaos.

A second, quite relevant, signature is the structure of the eigenstates themselves. Generally, the solutions of the time-independent Schrödinger equation have very nice mathematical properties, e.g. analyticity [51]; within its full delimited domain of validity, they are well behaved and represented by convergent series (for not too singular potentials). So, when we consider the lowest levels, for which the eigenstates do not possess many structures, usually we do not see a significant distinction between a regular and a chaotic system. Nevertheless, for higher eigenstates (not necessarily extremely high), the quantum manifestations of chaos become observable. For instance, the patterns of nodal lines can be very irregular [58], in comparison with those of the integrable case. Moreover, the 'chaotic' wave functions may present 'scars', as proposed by Heller [59]. For some eigenfunctions, the amplitudes are significant only around the spatial loci of unstable periodic orbits of the corresponding classical chaotic problem. This is particularly recurrent in billiard problems [60], a nice example being the scars of classical orbits for excited states of the stadium billiard [61].

Quantum chaos is not restricted to the above discussed aspects, other properties are also used to characterize quantum chaology. Although we are not going to discuss them here, for completeness, we just cite a few other identification possibilities: special features of wave-packet dynamics, the structure of avoided crossing (as we change parameters of the Hamiltonian), ergodicity in the wave-function state space; dynamical localization, etc.

As a last discussion in this section, we briefly comment on the very fundamental question of why a system that is nonlinear classically, when quantized should present particular features distinguishing it from the integrable case. In fact, this query does not have a definite answer, partially because we still do not know the precise border between the classical and quantum realms. A partial and rather speculative explanation has its foundations in an important point raised by Einstein [62] in 1917. Yet in the pre-QM era, the so-called Einstein–Brillouin–Keller (EBK) rules [51] already would lead to the (semiclassical) quantization of a problem in terms of invariant tori of the classical phase space. But such structures are possible only for integrable systems. Hence, Einstein pointed out that the quantization of chaotic dynamics was not possible in that way. Many years later, in breakthrough work, Gutzwiller derived his famous trace formula [51,63]. It associates the quantum spectra density of states with the periodic orbits of the classical problem. Without going into detail, the exact mathematical form of such an expression (in fact, of its pre-exponential amplitudes) is different if one has either a fully chaotic or a completely integrable classical dynamics [64]. So, the whole issue is that from a formal point of view, the semiclassical description in QM presents mathematical distinctions for the chaotic and integrable cases.

Let us now recall the correspondence principle. It states that below the Ehrenfest time (which again can belong to completely different scales for integrable and chaotic systems, typically being much shorter for the latter), somehow classical and QMs should be similar if $\hbar$ is small compared with other system parameters. But this is precisely one of the criteria for a good semiclassical description of QM. So, at least in such a limit condition, the classical distinctions between nonlinearity and regularity should find some sort of reciprocity in QM, which is just the type of distinction made in quantum chaos. Therefore, although very conjectural, the above remarks may give insights on why one could expect essential differences in the quantum dynamics of regular and nonlinear classical systems, despite the intrinsic linearity of the Schrödinger equation!

(*d*) *How does classical–quantum mixing generate nonlinear behaviour?*

We can summarize all the previous discussions as follows: (i) features associated with nonlinear behaviour can be present at nano and meso scales; they emerge either (ii) from nonlinear classical dynamics ruling distinct phenomena at such scales; or (iii) from pure QMs, in the context we call quantum chaos. From these observations, it should not be difficult to realize that at the classical–quantum border, manifestations of nonlinearities are actually possible.

As already mentioned, it is a very hard problem to determine the exact scale at which one switches from classical to quantum physics and how such a change takes place (for some interesting results on mesoscopic scales at very low temperatures, see [65]). Although much progress has been achieved [10], it is not yet a closed matter. So, instead of asking about the exact border between these two worlds we can be more 'pragmatic' and concentrate on the circumstances under which classical and quantum physics can mix, eventually giving rise to, e.g. chaos.

At the heart of this possibility is the effect of different sources non-coherently influencing the evolution of a given quantum subsystem. From a mathematical point of view, we can think of a quantum object possessing extra degrees of freedom (external or even internal) [66]. The subsystem itself is quantum, but if these extra degrees of freedom do not obey an usual quantum evolution, the interactions will make the quantum system depart from a pure quantum dynamics, possibly introducing stochasticity and (quantum noise) into the problem [67]. Therefore, we can have a mixed classical–quantum dynamics. A particularly important example is the Caldeira–Leggett model [68], which is very helpful to introduce dissipation in QM by means of a bath of harmonic oscillators. It leads to different interesting effects such as the suppression of tunnelling. These extra degrees of freedom (sometimes called environment) may behave classically because of their: size scales (being at or beyond the classical–quantum border), thermal origin, contact with further degrees of freedom, etc. In all these cases, the characteristics of the classical dynamics can induce nonlinear behaviour on the quantum subsystem [67,69].

Finally, the second situation is when the extra degrees of freedom (the interacting environment) are in essence quantum, but we either have only partial information on their features, or we have interest only in (or can only probe) the dynamics of the subsystem. As an important historical example of these 'open quantum systems', we mention polarons, discussed by R. P. Feynman with his path-integral approach. The idea is to describe the dynamics of an electron interacting with the phonons of a lattice. By making a trace over the phonon variables of the problem full propagator, one ends up with a dressed electron and thus can analyse the properties of its evolution alone. Similar to a classical environment, open quantum systems, depending on the nature of the interactions, can display very rich behaviour, even exhibiting features of quantum chaos [70].

3. The topics covered by this special issue

The present special issue, to which this piece is an introduction, is devoted to recent works related to nonlinearities in nano and meso scales. Naturally, it does not intended to exhaust a theme abundant in open questions, challenges and potential applications. Nevertheless, the idea is to put together a set of comprehensive papers addressing many different contexts where nonlinear behaviour emerges at very small scales, either setting the features of systems that are themselves of meso and nano sizes or being the smallest manifestation of nonlinearities in larger scales. The publication selects problems that are representative of the fundamental questions raised in the area, as well as of possible technological applications. In fact, by addressing topics such as:

— decoherence and the environment influence on quantum problems,
— large-molecule dynamics,
— fluid and plasma physics,
— nonlinear wave propagation,
— statistical mechanics and thermodynamics at small scales, and
— response theory,

this thematic issue can already serve as a 'guide' pointing to possible new directions of research, relevant problems that still need deeper analysis, and in what concrete situations, new methodologies and techniques should be developed to understand nonlinearity at very small scales.

In addition to the interest and relevance of the problems treated in the works in this special issue—of course, already properly presented and discussed in each individual paper—we would like to briefly highlight, throughout the brief comments below, some of their methodological strategies and technical protocols used to uncover the intricate behaviours observed in nonlinear systems at very short scales.

The idea of open quantum systems, namely those in which a quantum subsystem of interest interacts with extra (classical or quantum) degrees of freedom is addressed in the first four papers. They show how formally complicated problems can be treated in a consistent and physically meaningful way. Ozorio de Almeida & Brodier [71] describe how decoherence and dissipation can be studied through nonlinear semiclassical approximations using a chord-function representation. García-Mata *et al.* [72] analyse irreversibilities and sensitivities to (Hamiltonian) perturbations of quantum hyperbolic (strongly chaotic) maps coupled to the environment. Fluctuation theorems (usually associated with nonlinearities) in their quantum version are discussed by Campisi *et al.* [73], where an external work source sets the subsystem energy, measured in terms of the differences in the subsystem energies at the beginning and end of the process (the so called exclusive viewpoint). Finally, within this topic, da Silva *et al.* [74] address technologically relevant microscopic semiconductor devices.

They analyse, by means of microscopic Poisson processes, how the presence of noise (e.g. owing to thermally activated random traps) induces nonlinearity in the system current responses.

The next three works deal with nonlinear behaviour in large molecular systems, which, although lying in scale sizes of the order of 10^{-8}–10^{-6} m, can be described by classical physics. In particular, these papers emphasize the important role that the environment has in inducing complex and nonlinear behaviour at such scales. Tellez [75] discusses charged macromolecules interacting with a dispersion medium, formed by much smaller molecules, therefore treated as an effective potential in a mean-field approximation. The interaction of macromolecules with the dispersion medium induces a nonlinear screening of charges, thus giving rise to interesting phenomena. Pezzuti *et al.* [76] consider another problem of technological interest, the long-range organization of block copolymers, but which locally assemble at a nano-scale architecture. The nonlinear dynamics of topological defects in these extended structures determines the system relaxation properties. Marconi *et al.* [77] analyse the complex process of string-gel formation by dipolar colloidal particles, showing from molecular dynamics that more simple nucleation mechanisms cannot explain the phases observed in such systems.

Fluid dynamics provides paradigmatic examples of nonlinear and complex behaviour. Indeed it is described, in the more general case, by nonlinear equations (e.g. the Navier–Stokes equation), and can present extreme phenomena such as turbulence. In two works of this issue, it is shown that the characteristic sizes in which nonlinearities may arise in fluid and plasma physics can be in the nano and meso scales. In the son–father collaboration, Viswanathan & Viswanathan [78] tackle the extremely hard problem of stability and solvability of the Navier–Stokes equation. They argue that the difficulty found in settling the question has its roots in the interplay between the nonlinearity and the scale-invariant aspects (which can cascade to very small scales) in hydrodynamics. Viana *et al.* [79] discuss a topic of applied interest: nonlinear plasma dynamics and its emerging fractal structures. Such structures are fundamental in determining the transport properties, e.g. in tokamaks. The work emphasizes how fractality is an important ingredient to understand meso-scale processes in plasmas, e.g. electromagnetic turbulence.

Many different phenomena are inherently caused by non-equilibrium dynamics and long-range interactions, which in their turn have a nonlinear origin. Furthermore, as discussed in the present introductory paper, nonlinearity may equally well occur at very large and at very small scales. As examples, we can cite surface growth, wave propagation, pattern formation and anomalous relaxation (due to quasi stationary states). In four very interesting papers, different theoretical approaches (based on distinct mathematical methods) are used to deal with such diverse contexts. Wio *et al.* [80] use the idea of non-equilibrium potentials and a stochastic nonlinear partial differential equation, known as Kardar–Parisi–Zhang, to phenomenologically discuss surface and interface growth in mesoscopic systems. The effect of spatial discreteness—for instance, present in atomic lattices—on the wave-front propagation that is under the action of damping is described by Clerc *et al.* [81] by means of continuous differential equations. Masoller & Rosso [82] consider a way to quantify statistical complexity (useful in characterizing pattern formation in quantum nano and meso problems) for a prototype system, the (delayed) logistic map with nonlinear feedback. Also, Ruffo *et al.* [83] address the Vlasov equation for uncoupled pendula (analogous to a model used in QM to treat transport in semiconductor nanodevices), which can describe the kinetic behaviour of long-range interacting systems (mainly of electrical origin).

Einstein once said that '[Thermodynamics] is the only physical theory of universal content concerning which I am convinced that, within the framework of applicability of its basic concepts, it will never be overthrown'. If we extend this comment to the more general and system-independent aspects of (especially equilibrium) statistical mechanics, we should not expect the necessity of any modification of such theories when addressing nonlinear systems. Nevertheless, considering that in some cases either: (i) the very small scales are almost outside the scope of thermodynamics or (ii) the particular features of nonlinearity eventually would lead to extra ingredients helpful to any analysis; then, it might be the case for statistical mechanics descriptions to explicitly take into account the specificities of nonlinear behaviour at small scales. Along this line, Beck [84] discusses why superstatistics (i.e. superposition of several statistics at different time scales) are useful in describing mesoscopic systems at slowly fluctuating environments. Also, the work shows how superstatistics can be mapped to the ordinary case, but with modified energy levels. Benenti & Casati [85] review an idea of potentially great practical application. By considering the thermodynamics of nonlinear dynamical systems, they address a long-standing problem of increasing the efficiency of thermoelectric machines, proposing interesting new insights into the problem.

As the last contribution and somehow within the general framework of response theory, Ruelle [86] discusses a physical measure (known as Sinai–Ruelle–Bowen, SRB) for some relevant quantity (a smooth function) over a discrete time evolution system, a map. By associating a susceptibility (very closely related to, but slightly different from, the usual definition in physics) to the above-mentioned quantity, one can ask when the susceptibility can be calculated. In other words, when it is possible to obtain a meaningful expression with no singularities, thus giving relevant information about the dynamical problem at hand. The work analyses the mathematical stability of the susceptibility according to the features of the dynamical system, namely the presence, or not, of stable and unstable manifolds in the phase space. The results highlight the formal cautions necessary in (theoretically) studying nonlinear systems.

4. Conclusion

Given the general purpose of this special issue, the selected papers cover distinct questions under the focus of present interest in nonlinear systems at small scales. Actually, as made clear in §§1 and 2, there are many different physical contexts where nonlinearities can be manifested in typical sizes of the order of 10^{-9}–10^{-6} m. Furthermore, from a formal point of view, they can be treated within diverse frameworks, depending on the nature of the problem. Hence, the articles constituting this special issue address the most common techniques and theoretical approaches in the field, also discussing many of the huge number of phenomena emerging from nonlinearities at such scales. So, this survey may help the reader to gain insights into fundamental questions as well as into methodological tools necessary to tackle the study of nonlinearities in nanoscopic and mesoscopic systems. Moreover, it might serve as a 'guide' pointing to possible new directions of research, relevant theoretical and applied problems that still need deeper analysis, and what kind of new concepts are necessary to understand and exploit nonlinearity at the quantum and quantum–classical realms.

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