Carnot cycle for interacting particles in the absence of thermal noise

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A thermodynamic formalism is developed for a system of interacting particles under overdamped motion, which has been recently analyzed within the framework of nonextensive statistical mechanics. It amounts to expressing the interaction energy of the system in terms of a temperature θ, conjugated to a generalized entropy s_q, with $q = 2$. Since θ assumes much higher values than those of typical room temperatures $T \ll \theta$, the thermal noise can be neglected for this system ($T/\theta \simeq 0$). This framework is now extended by the introduction of a work term δW which, together with the formerly defined heat contribution ($\delta Q = \theta ds_q$), allows for the statement of a proper energy conservation law that is analogous to the first law of thermodynamics. These definitions lead to the derivation of an equation of state and to the characterization of s_q adiabatic and θ isothermic transformations. On this basis, a Carnot cycle is constructed, whose efficiency is shown to be $\eta = 1 - (\theta_2/\theta_1)$, where θ_1 and θ_2 are the effective temperatures of the two isothermic transformations, with $\theta_1 > \theta_2$. The results for a generalized thermodynamic description of this system open the possibility for further physical consequences, like the realization of a thermal engine based on energy exchanges gauged by the temperature θ.

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I. INTRODUCTION

Thermodynamics is a macroscopic theory based on empirical laws leading to several physical and technological applications, among which one can emphasize thermal engines [1,2]. Among many possible engines, the most celebrated is the one described by the Carnot cycle. It played a fundamental role in the historical and theoretical development of thermodynamics by providing an operational method for measuring the ratio of two temperatures based on the concept of cycle efficiency. As a consequence, the choice of a reference system, on which one arbitrarily imposes a temperature value, allows the definition of a universal temperature scale, up to a multiplicative constant. The ability for temperature measurements opens the possibility of entropy variation (or relative entropy) estimates. The Nernst postulate turns such relative entropies into absolute entropies, so that one becomes able to evaluate the entropy of a given physical system, providing a complete thermodynamic description.

The recent analysis of a model for the overdamped motion of interacting vortices in a type-II superconductor [3] within the framework of nonextensive statistical mechanics has been quite successful in describing the behavior of this system in terms of an effective temperature θ, which is thermodynamically conjugated to a generalized entropy per particle, s_q, with $q = 2$, to be called hereafter s_2. While θ bears no resemblance to the usual temperature T, its definition

$$k\theta = \frac{N\pi f_0 \lambda^2}{L_y} = n\pi f_0 \lambda^2 \tag{1}$$

can be easily interpreted in terms of certain physical properties of the system. In the equation above k stands for the Boltzmann constant, λ is the London penetration length, and f_0 represents the interaction strength, to be defined more precisely later on. One sees that θ is directly related to $n = N/L_y$, i.e., the number of vortices N per unit of length L_y. As will be discussed in the next section, it is possible to vary n experimentally, while the other quantities appearing in Eq. (1) are fixed for each type-II superconductor. It should be emphasized that this temperature is associated with the interaction among vortices, as well as their density, but not with their kinetic energy. We also recall that the quoted analysis allowed for the derivation of several relations among the system parameters and variables, which are all in all similar to those of standard thermodynamics, e.g., just by replacing $T > 0$ by $\theta > 0$.

The purpose of this work is to extend the previous investigations by showing that other concepts in the usual thermodynamics have their natural counterparts within a generalized thermodynamics that is related to nonextensive statistical mechanics. In particular, we show that a proper identification of a generalized work term δW and the use of the previously introduced heat term $\delta Q = \theta ds_2$ lead to the recovery of Carnot's results for a machine operating between two heat reservoirs (these are, within this proposal, reservoirs of vortices characterized by different densities). We emphasize that the relevant concept for energy exchange in the current approach is that of the "vortex reservoir," a system containing a much larger number of vortices than the one under study. As will be detailed later on, the reservoir temperature θ is not altered when it receives or delivers energy from or to the considered system.

The paper is organized as follows: Sec. II briefly describes some relevant experimental features of the type-II superconductor vortex system. In the following section we discuss a nonlinear Fokker-Planck equation (NLFPE) that has been derived directly by means of a coarse-graining approach applied to the microscopic equations of motion of the corresponding interacting-vortex model [4–6]. Next we characterize, in Sec. IV, the equilibrium states and discuss

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the appropriateness of the effective-temperature definition introduced in Ref. [3]. Furthermore, we introduce a definition of an infinitesimal work term for this system, δW, which is suitable to be combined with the heat term δQ in order to state a law analogous to the first law of thermodynamics in its infinitesimal form. This major contribution of the present study leads to an equation of state and to a further physical interpretation of the effective temperature, in terms of the variance of the vortex positions, $\theta \propto \langle x^2\rangle^{3/2}$. In Sec. V we indicate how two specific energy exchange processes can be carried out, namely, the s_2-adiabatic and θ-isothermic transformations. The latter requires the system to be in contact with a "vortex heat reservoir," a concept that is also introduced in this section. Finally, these definitions allow the construction of a Carnot cycle, for which the efficiency is shown to be given by $\eta = 1 - (\theta_2/\theta_1)$, where θ_1 and θ_2 (with $\theta_1 > \theta_2$) are the temperatures of two vortex reservoirs. This result strongly supports the previous definition of the effective temperature θ and of a consistent thermodynamic framework for this physical system. Finally, Sec. VI closes the work with our concluding remarks.

II. THE SUPERCONDUCTOR VORTEX SYSTEM

In type-II superconductors, vortices are generated by an external magnetic field B_{ext}, whose lines become confined to flux tubes [7] and, for a given material, the quantities f_0 and λ (introduced in the previous section) present well-defined values. So, according to Eq. (1), any proposal for varying θ experimentally should be directly related to an adequate tuning of the vortex density n. In fact, recent experimental researches in this area led to considerable advances in the ability to control many properties of these vortices, like their motion and density [8–11]. In particular, the density n may be varied either by changing B_{ext} appropriately (typical estimates in this case are given in [3]), or by applying an alternating electrical current [8–11]. Within this latter procedure, one may even eliminate all vortices, yielding the desirable limit $n \to 0$. All these aspects can be adequately described with the concept of the effective temperature θ advanced in the previous section, including the limit $\theta \to 0$.

The variable θ has been sometimes referred to as an "effective temperature" [3]. This name has been widely used in physics to represent the typical scale of energy per particle of a given system. Therefore, it gauges the effect of typical thermal fluctuations in the system as compared to the mechanical and electromagnetic properties of its particles, such as their masses, spin, harmonic frequencies, and concentration. As an example, one might mention the Fermi temperature (T_F) in a Fermi-Dirac ideal gas, which is directly related to the concentration of electrons and results from the analysis of the system at $T = 0$. Thus, it presents well-defined values for a given Fermi gas, e.g., $T_F \approx 10^4$ K for electrons in metals [1,2], or $T_F \approx 10^9$ K for electrons in white dwarf stars [2]. In real physical situations, one may treat electrons in metals as a Fermi-Dirac ideal gas for experimental realizations carried out at temperatures $T \ll T_F$, a condition that is satisfied even at room temperatures.

Compared to the usual effective-temperature concepts in the literature, the difference in Eq. (1) is that it does not simply provide an energy scale for the system, but is associated with its state and may be varied experimentally by changing the density of vortices ($\theta \propto n$). The importance of associating θ with effective-temperature concepts concerns the relation to other features of the system (different from those of the usual temperature T), and also that it attains much larger values than room temperatures. Analogously to the Fermi temperature in a Fermi-Dirac ideal gas, the effective temperature defined in Eq. (1) appears in the theoretical description on considering the vortex system at relatively low temperatures $(T/\theta) \simeq 0$, in which case the Boltzmann-Gibbs entropy may be neglected.

Typical values of θ in type-II superconductors were estimated recently to lie in the range $10^8 \to 10^{12}$ K [3]. Some of these estimates appear to be extremely high (even when compared with the Fermi temperatures of electrons in white dwarf stars), but one should keep in mind that θ represents an effective temperature associated with the interaction among vortices. Based on this, experimental investigations of physical properties associated with θ may be performed in a temperature range that is limited only by the existence of a superconducting phase. From the theoretical point of view, the system is well approximated by a model satisfying the condition $(T/\theta) \simeq 0$, where the effects of thermal noise can be neglected. Moreover, the product $k\theta$ introduced in Eq. (1) presents the dimension of energy, is positive definite, and is directly related to the density $n = N/L_y$, as well as to the interaction among vortices, f_0 and λ. This energy was also estimated in Ref. [3], leading to very high values, in the range $10^{-14} \to 10^{-11}$ J, comparable to the rest energy of the electron (8.19×10^{-14} J) and other known particles. Therefore, it would be most desirable to use this type of system to perform work. This represents one of the motivations of the present investigation.

III. THE MODEL AND ASSOCIATED ENTROPY

The temperature definition in Eq. (1) emerged through its association with the diffusion coefficient D ($D \equiv k\theta$) in the following NLFPE [4–6], which accurately describes the vortex system:

$$\mu \frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + 2D\frac{\partial}{\partial x}\left\{[\lambda P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}. \quad (2)$$

This association has its justification in the similar procedure that is usually adopted in the framework of the linear Fokker-Planck equation [12]. Moreover, this equation depicts the effects of an external potential $\phi(x)$ [$A(x) = -d\phi(x)/dx$], as well as of $N - 1$ vortices, on a tagged vortex. Therefore, the distribution $P(x,t)$ refers to one vortex of the above-mentioned system and consequently all physical quantities to be derived from this distribution will correspond to one-vortex properties.

The above NLFPE was obtained from a coarse-graining approximation of the equations of motion of N repulsively interacting vortices, under overdamped motion, in a rectangular box of side lengths L_x and L_y, in a medium with an effective friction coefficient μ,

$$\mu \mathbf{v}_i = \frac{f_0}{2} \sum_{j \neq i} K_1(r_{ij}/\lambda)\, \hat{\mathbf{r}}_{ij} + \mathbf{F}_i^{\rm ext}. \quad (3)$$

In the above equation, $\mathbf{v}_i$ represents the velocity of vortex i, and the terms on the right-hand side depict the forces acting on vortex i $(i = 1,2,\ldots,N)$. The first contribution takes into account the interactions among vortices [each vortex interacts with the remaining $(N-1)$ vortices], whereas $\mathbf{F}^{\rm ext} = -A(x)\hat{\mathbf{x}}$ represents an external force acting on vortex i. The vortex-vortex interactions are repulsive and radially symmetric, expressed in terms of a modified Bessel function of the second kind of order 1, $K_1(r_{ij}/\lambda)$, $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ stands for the distance between vortices i and j, and $\hat{\mathbf{r}}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)/r_{ij}$ is a unit vector defined along the axis joining them. Moreover, f_0 is a positive constant, known as the pinning strength, characteristic of each physical system, whereas the Bessel function decays according to a characteristic length scale λ, the so-called London penetration length [7,13–15]. These repulsive forces are opposed by an external confining potential in the x direction, which herein we consider as a harmonic one, i.e., $A(x) = -\alpha x$ $(\alpha > 0)$, acting on each vortex. Therefore, by starting the simulations with all particles confined in a small region around $x = 0$, the repulsive vortex-vortex interactions prevail at the initial times, with the vortices moving apart quickly, until the external potential becomes significant, so that the system reaches a stationary state after some sufficiently long time [5,6]. For simplicity, the box size L_x is chosen sufficiently large such that the vortices never reach the walls in the x direction, whereas periodic boundary conditions are considered in the y direction.

It is important to mention that the physical system defined in terms of interacting vortices of Eq. (3) has been much used in the literature to model magnetic flux lines in disordered type-II superconductors (see, e.g., Refs. [4,7,13–15]). Besides the coarse-graining approximation, which led to Eq. (2), the connection of the present physical system with this equation was further supported by a remarkable agreement found between the vortex-position stationary-state [5] and time-dependent [6] distributions, obtained by means of molecular-dynamics simulations and the analytical solution of the NLFPE [16,17].

NLFPEs have been widely investigated [18], motivated by the search for an appropriate description of many complex physical systems. A particular interest has been dedicated to the NLFPE proposed in Refs. [16,17], related to Tsallis nonextensive statistical mechanics [19–22]. In particular, the q-Gaussian distribution, which represents a generalization of the standard Gaussian (recovered in the particular case $q = 1$), appears naturally from an extremization procedure of the entropy [22], or from the solution of the corresponding NLFPE of Refs. [16,17], and it has been very useful for experiments in many real systems [19–21]. Moreover, similarly to standard procedures used for the linear Fokker-Planck equation [12], proofs of the H theorem have been achieved recently by considering nonlinear Fokker-Planck equations. Therefore, in the same way that the linear Fokker-Planck equation is associated with normal diffusion and with the Boltzmann-Gibbs entropy, the NLFPEs are usually related to anomalous-diffusion phenomena and to generalized entropies, by means of the H theorem (see, e.g., Refs. [18,23–28] among others).

In the present case, the pertinent H theorem may be proved by imposing a well-defined sign for the time derivative of the free-energy functional per vortex [5,6],

$$f = u - \theta s_2, \quad u = \int_{-\bar{x}(t)}^{\bar{x}(t)} dx\, \phi(x) P(x,t), \quad (4)$$

and making use of Eq. (2). In the above free energy, u represents the one-particle internal energy, $\phi(x)$ depicts the external potential introduced in Eq. (2), and θ corresponds to the effective temperature defined in Eq. (1). Moreover, the distribution $P(x,t)$ presents a compact support, $-\bar{x}(t) \leqslant x(t) \leqslant \bar{x}(t)$. In order to satisfy the H theorem, the associated entropy should be given by [5,6]

$$s_2[P] = k \left\{ 1 - \lambda \int_{-\bar{x}(t)}^{\bar{x}(t)} dx\, [P(x,t)]^2 \right\}. \quad (5)$$

From now on, we will be restricted to the long-time limit, for which one attains the stationary-state solution $\lim_{t\to\infty} P(x,t) = P_{\rm st}(x)$, defined in terms of the finite support $\lim_{t\to\infty} \bar{x}(t) = x_e$. As verified in the numerical simulations, in this limit the system is characterized by a well-defined equilibrium state [5,6]; therefore, the above-mentioned stationary-state solution corresponds to this equilibrium state. Moreover, the H theorem guarantees that the system will always approach this state for sufficiently long times. This equilibrium state will be considered now as the basis for our thermodynamic framework.

IV. EQUILIBRIUM STATE AND FIRST LAW

The stationary-state solution of Eq. (2) is [5,6]

$$P_{\rm st}(x) = \frac{\alpha}{4k\theta\lambda}\left(x_e^2 - x^2\right) = \frac{\alpha\lambda}{4k\theta}\left[\left(\frac{x_e}{\lambda}\right)^2 - \left(\frac{x}{\lambda}\right)^2\right], \quad (6)$$

with $|x| < x_e$, where $x_e = (3k\theta\lambda/\alpha)^{1/3}$ is found from the normalization condition for $P_{\rm st}(x)$. As expected from a proper temperature parameter, the width of $P_{\rm st}(x)$ increases as θ gets larger [3]. In addition to this, for a fixed θ, the volume occupied by the particles, $2x_eL_y$, decreases for increasing values of α.

From the above distribution one may calculate physical quantities at the stationary state, like the entropy and internal energy. As shown in Refs. [3,5,6], in order to fulfill the H theorem, the appropriate functional forms for these quantities should be given respectively by Eqs. (5) and (4), leading to

$$\frac{s_2}{k} = 1 - \lambda \int_{-x_e}^{x_e} dx\, [P_{\rm st}(x)]^2 = 1 - \frac{3^{2/3}}{5}\left(\frac{\alpha\lambda^2}{k\theta}\right)^{1/3}, \quad (7)$$

$$u = \int_{-x_e}^{x_e} dx\, \frac{\alpha x^2}{2} P_{\rm st}(x) = \frac{3^{2/3}}{10}(\alpha\lambda^2)^{1/3}(k\theta)^{2/3}. \quad (8)$$

After adequate manipulation of the above quantities, the entropy may be expressed in terms of the internal energy,

$$s_2(u,\alpha) = k\left[1 - \frac{3}{5}\left(\frac{\alpha\lambda^2}{10u}\right)^{1/2}\right], \quad (9)$$

or equivalently, the internal energy in terms of the entropy,

$$u(s_2,\alpha) = \frac{9}{250}\,\frac{\alpha\lambda^2}{(1 - s_2/k)^2}. \quad (10)$$

One should notice that in Eqs. (9) and (10) we have explicitly written a dependence on the parameter α, which, as already discussed, corresponds to an external parameter associated with the confinement of vortices. From these equations one obtains the fundamental relations

$$\left(\frac{\partial s_2}{\partial u}\right)_\alpha = \frac{1}{\theta}, \quad \left(\frac{\partial u}{\partial s_2}\right)_\alpha = \theta, \quad (11)$$

showing the suitability of the definition introduced in Eq. (1). Moreover, a specific-heat-like quantity has been defined for fixed α, satisfying

$$c_\alpha = \left(\frac{\partial u}{\partial \theta}\right)_\alpha = \theta\left(\frac{\partial s_2}{\partial \theta}\right)_\alpha, \quad (12)$$

suggesting the definition of an infinitesimal amount of heatlike contribution to the energy change, $\delta Q = \theta ds_2$.

As usual, the work contribution should come from the external potential, which in the present case corresponds to the parameter α, directly related to the volume occupied by the vortices in the stationary state. From this external potential acting on each particle, we define heuristically the infinitesimal work as $\delta W = \sigma d\alpha$, where σ represents a parameter conjugated to α (with dimensions $[\sigma] = L^2$), to be determined later on. Considering these definitions, an equivalent to the first law becomes

$$du = \delta Q + \delta W = \theta ds_2 + \sigma d\alpha, \quad (13)$$

where δW corresponds to the work done *on* the system. The equation of state $\sigma = \sigma(\theta,\alpha)$ may be obtained by noticing that Eq. (13) yields $(\partial s_2/\partial\alpha)_u = -\sigma/\theta$, whereas deriving Eq. (9) and using the internal energy of Eq. (8), one obtains the following equation of state:

$$\sigma = \frac{3^{2/3}}{10}\,\lambda^2\left(\frac{k\theta}{\alpha\lambda^2}\right)^{2/3} \Rightarrow \sigma = \frac{u}{\alpha}. \quad (14)$$

The relation above ($\sigma\alpha = u$) involving the two conjugated parameters associated with the infinitesimal amount of work δW and the internal energy u may be compared with the one for an ideal gas, namely, $pv = 2u/3$ (valid for the classic case, as well as in both types of quantum statistics [12]).

From the definition of internal energy in Eq. (8) and considering the result of Ref. [3], $\langle x^2\rangle = x_e^2/5$, one obtains

$$u = \tfrac{1}{10}\,\alpha x_e^2 \Rightarrow \sigma = \tfrac{1}{10}\,x_e^2 = \tfrac{1}{2}\,\langle x^2\rangle, \quad (15)$$

which implies that σ is a non-negative quantity. Dealing with Eqs. (14) and (15) one obtains

$$k\theta = \frac{5^{3/2}}{3}\,\frac{\alpha}{\lambda}\,\langle x^2\rangle^{3/2}. \quad (16)$$

This result provides an interpretation for the effective temperature θ, which is herein related to the particle-position deviation, defined according to the distribution $P_{\rm st}(x)$ of Eq. (6). Therefore, similarly to the concept of the kinetic temperature of a classical gas, for which the temperature is related to the second moment of the corresponding velocity probability distribution, i.e., $T \propto \langle v^2\rangle$, in the present case one has $\theta \propto \langle x^2\rangle^{3/2}$.

Further remarks follow from the relations above: (i) Eq. (15) expresses a relation between σ and the width of the distribution $P_{\rm st}(x)$. In this way, one sees that the work term $\delta W = \sigma d\alpha$ acts directly on the stationary distribution, e.g., for fixed θ, a positive δW will reduce x_e, decreasing the width of $P_{\rm st}(x)$. (ii) The equation of state [Eq. (14)] implies that the parameter σ introduced in Eq. (13) increases with θ (for fixed α), whereas for fixed θ, an increase in σ yields a decrease in α. (iii) The results above suggest a correspondence of the parameters introduced in Eq. (13) with those of an ideal gas in standard thermodynamics: $(\sigma,\alpha^{-1},\theta) \Leftrightarrow (p,v,T)$.

V. PHYSICAL TRANSFORMATIONS AND THE CARNOT CYCLE

Let us now address some simple physical transformations. First, we will introduce the concept of a vortex heat reservoir $\mathcal{R}$, which is defined as a system containing a much higher number of vortices than the system $\mathcal{S}$ under study. The concept of reservoir plays an important role in thermodynamics and statistical mechanics [1,2,12]. Within thermodynamics, a reservoir defines a given equilibrium property of the system which it interacts with, like its temperature (in the case of a heat reservoir), its chemical potential (in the case of a particle reservoir), or its pressure (in the case of a volume reservoir). In statistical mechanics, a proper choice of the ensemble (e.g., canonical, grand-canonical, or pressure ensemble) depends essentially on the type of system-reservoir interaction.

Let us now indicate how a contact between these two systems $\mathcal{R}$ and $\mathcal{S}$ should occur, in such a way as to produce effective-temperature changes. From Eq. (1) one has $\theta \propto n$, i.e., $\theta \propto N/L_y$, so that temperature variations are associated with changes in N or L_y (or both). Herein, we will restrict ourselves to θ variations, keeping the total number of vortices N fixed, corresponding to alterations in the length L_y. Since the vortices interact repulsively, they should produce a pressure on the horizontal wall separating the two systems. Hence, a change in θ may be attained through the displacement of a nonpermeable horizontal wall separating the systems $\mathcal{R}$ and $\mathcal{S}$, so that a variation δL_y becomes negligible for the larger system $\mathcal{R}$, because $\mathcal{R}$ is much larger than $\mathcal{S}$.

Thus, when the vortex heat reservoir is put in contact with the smaller system, an out-of-equilibrium situation is observed if their temperatures $\theta_{\mathcal{R}}$ and $\theta_{\mathcal{S}}$ are different. The above-mentioned interaction between the two systems will occur until their densities $n_{\mathcal{R}}$ and $n_{\mathcal{S}}$ become equal, keeping constant the number of vortices in both systems. Since $\mathcal{R}$ is much larger than $\mathcal{S}$, the vortex heat transfer does not change significantly the density of vortices of the reservoir. After some time, an equilibrium situation is reached for which the larger system will define the density of vortices of the smaller one, and consequently its effective temperature $\theta_{\mathcal{S}} = \theta_{\mathcal{R}}$. In what follows, the isothermal transformations to be considered are presumed to occur for a system of vortices in contact with such reservoirs.

Moreover, we recall that considering the expressions for the entropy in Eqs. (7) and (9), as well as the relation $\sigma = u/\alpha$, one concludes that an adiabatic process corresponds to one of the conditions

$$\frac{\alpha}{\theta} = \text{const}, \quad \sigma = \frac{u}{\alpha} = \text{const}, \tag{17}$$

or properly defined combinations of them. Therefore, the total work done *on* the system in an adiabatic transformation from an initial state characterized by (θ_i,α_i) to a final state with (θ_f,α_f) will be given by

$$u_f - u_i = W = \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \sigma(\alpha_f - \alpha_i), \tag{18}$$

where σ is given by Eq. (14), with $(\theta/\alpha) = $ const. Hence, in an adiabatic process one gets positive (negative) work for $\alpha_f > \alpha_i$ ($\alpha_f < \alpha_i$). One should notice that in a plot of σ versus α the adiabatic transformation is represented by a horizontal line, and the area below this line corresponds to the work done.

For an isothermal process at a temperature θ, one has

$$Q = \int_{s_{2,i}}^{s_{2,f}} \theta ds_2 = \frac{3^{2/3}}{5}(k\theta\lambda)^{2/3}\left(\alpha_i{}^{1/3} - \alpha_f{}^{1/3}\right), \tag{19}$$

$$W = \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \frac{3^{5/3}}{10}(k\theta\lambda)^{2/3}\left(\alpha_f{}^{1/3} - \alpha_i{}^{1/3}\right), \tag{20}$$

$$u_f - u_i = Q + W = \frac{3^{2/3}}{10}(k\theta\lambda)^{2/3}\left(\alpha_f{}^{1/3} - \alpha_i{}^{1/3}\right), \tag{21}$$

where the above internal energy variation may be calculated either directly from Eq. (8), or by $u_f - u_i = Q + W$, as expected. Therefore, for the isothermal process one has positive (negative) work and variation of internal energy, whereas the system releases (absorbs) heat if $\alpha_f > \alpha_i$ ($\alpha_f < \alpha_i$).

As mentioned above, the values of $k\theta$ for typical type-II superconductors may be very high, when compared to standard thermal energies. Moreover, the stationary-state solution $P_{\text{st}}(x)$ of Eq. (6) is expected to occur for $k\theta \sim \alpha\lambda^2$. Since the energies of Eqs. (18)–(21) all depend on differences involving the final and initial values of the vortex-confining parameter α, considerable amounts of energy may appear if one is able to perform thermodynamic transformations characterized by an expressive variation in these parameters. Although this remains as an experimental task, a large amount of work might be obtained in an isothermal transformation like the one of Eq. (20), if one could get a significant difference $(\alpha_f{}^{1/3} - \alpha_i{}^{1/3})$ throughout its experimental realization.

From the transformations above, one can define a cycle analogous to the Carnot cycle, by considering two isothermal and two adiabatic processes, intercalated, as illustrated in Fig. 1 in the plane σ/λ^2 (dimensionless) versus $\alpha\lambda^2$ (dimensions of energy); some of its properties are listed next. (i) An amount of heat Q_1 is absorbed in the isothermal process at the higher temperature θ_1, whereas the system releases heat Q_2 in the isothermal process at the lower temperature θ_2. (ii) In a plot of σ versus α (or equivalently, σ/λ^2 versus $\alpha\lambda^2$, as in Fig. 1), the work associated with a given process corresponds to the area below this transformation. As shown above, work is positive (negative) for transformations that increase (decrease) α. Therefore, the total work done *on* the system, calculated as $W = W_{ab} + W_{bc} + W_{cd} + W_{da}$, is given by the area enclosed in the cycle of Fig. 1, and is negative, as expected from Eq. (13). If one defines $\mathcal{W} = -W$ as the work done *by* the system, the variation of internal energy is zero for the complete cycle, and one has $Q_1 = \mathcal{W} + Q_2$ (conventionalizing all these three quantities as positive). (iii) By manipulating Eqs. (14) and (19), one obtains the well-known result relating the two isothermal processes, $(Q_1/Q_2) = (\theta_1/\theta_2)$, leading to the celebrated efficiency of the Carnot cycle,

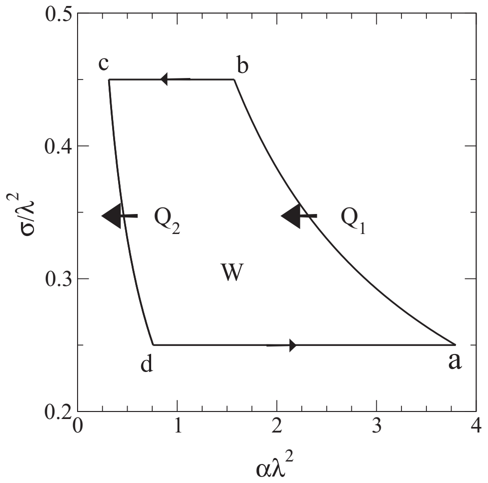

FIG. 1. The Carnot cycle $a \to b \to c \to d \to a$, for a system of interacting vortices under overdamped motion at $(T/\theta) \simeq 0$. The transformations for constant σ are adiabatic, and herein they were chosen to occur for $(\sigma/\lambda^2) = 0.45$ ($b \to c$) and $(\sigma/\lambda^2) = 0.25$ ($d \to a$). The isothermal transformations are characterized by $\sigma \sim \alpha^{-2/3}$ [cf. Eq. (14)] and they occur for $k\theta_1 = 5$ (units of energy) in $a \to b$, and $k\theta_2 = 1$ (units of energy) in $c \to d$, i.e., $\theta_1 > \theta_2$. The area inside the cycle represents the total work W done *on* the system, which is negative, as expected from Eq. (13). The abscissa $\alpha\lambda^2$ presents dimensions of energy, whereas the ordinate σ/λ^2 is dimensionless; the cycle above holds for any system of units, e.g., one may consider all quantities with dimensions of energy in joules.

$$\eta = \frac{\mathcal{W}}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{\theta_2}{\theta_1} \quad (0 \leqslant \eta \leqslant 1). \tag{22}$$

This result provides a strong support for the idea that the fundamental relation considered herein, as analogous to the first law of thermodynamics [cf. Eq. (13)], is appropriate for this system.

VI. CONCLUSIONS

To conclude, we have introduced concepts, analogous to those of standard thermodynamics, for a system of interacting vortices under overdamped motion, in which thermal noise can be neglected [$(T/\theta) \simeq 0$]. First of all, the important idea of an effective temperature θ, proposed in a previous work [3], which is related to the density as well as to the interactions among vortices (being always positive, by definition), was investigated more deeply herein.

In this study, we have introduced a work term δW, in order to formulate an infinitesimal form for the first law. From this proposal, an equation of state follows naturally; the first law, together with the equation of state, is a fundamental requisite for defining physical transformations, and consequently leading to the possibility of physical realizations. Moreover, using this equation of state we have found a physical interpretation for the effective temperature, which is associated with the variance in the particle positions, namely, $\theta \propto \langle x^2\rangle^{3/2}$. This should be compared with the interpretation of the kinetic temperature in a classical gas, $T \propto \langle v^2\rangle$.

Considering simple transformations, like adiabatic and isothermal processes, we have constructed a cycle analogous to the Carnot cycle, and have shown that its efficiency is given by $\eta = 1 - (\theta_2/\theta_1)$, where θ_1 and θ_2 represent the effective temperatures associated with the "hotter" (higher vortex density) and "colder" (lower vortex density) heat reservoirs, respectively, showing the appropriateness of the definition of θ. Therefore, we have presented a consistent thermodynamiclike framework, giving further support for the system considered herein as an important physical application for nonextensive statistical mechanics.

Finally, one should emphasize the following advantages of the engine proposed herein, with respect to standard thermal machines: (i) Significant amounts of work may result from some thermodynamic transformations proposed. (ii) In standard thermal engines one usually comes across the undesirable effect of overheating of the machine, due to exchanges of heat between the engine and its environment. In the engine proposed herein changes of the effective temperature are solely associated with changes in the concentration of vortices, and as a consequence, overheating is not expected to occur. (iii) From the operational point of view, since typical values of θ in real systems are much higher than room temperature, one expects that thermal effects should not affect experiments carried out at typical superconducting temperatures. Consequently, measurements in type-II superconductors, under an appropriate confining potential, turn out to be highly desirable in order to confirm the validity of these theoretical predictions.

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