Impact of site dilution and agent diffusion on the critical behavior of the majority-vote model

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In this work we study a modified version of the majority-vote model with noise. In particular, we consider a random diluted square lattice wherein a site is empty with a probability r. In order to analyze the critical behavior of the model, we perform Monte Carlo simulations on lattices with linear sizes up to $L = 140$. By means of a finite-size scaling analysis we estimate the critical noises q_c and the critical ratios β/ν, γ/ν, and $1/\nu$ for some values of the probability r. Our results suggest that the critical exponents are different from those of the original model ($r = 0$), but they are r independent ($r > 0$). In addition, if we consider that agents can diffuse through the lattice, the exponents remain the same, suggesting a new universality class for the majority-vote model with noise. Based on the numerical data, we may conjecture that the values of the exponents in this universality class are $\beta \sim 0.45$, $\gamma \sim 1.1$, and $\nu \sim 1.0$, which satisfy the scaling relation $2\beta + \gamma = d\nu = 2$.

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I. INTRODUCTION

Social dynamics have been studied through statistical physics techniques in the last twenty years. Among the studied problems, we highlight a series of works dealing with opinion dynamics [1–13] (for a recent review, see Ref. [14]). These models are interesting for physicists because they present order-disorder transitions, scaling, and universality among other typical features of physical systems [14].

Concerning models of opinion dynamics, one of the most studied systems is the majority-vote model with noise [15]. In the original model, each site of an isotropic square lattice has an Ising-spin variable whose state ± 1 may be associated with the opinion of an individual in a social community. The time evolution of the model is governed by inflow dynamics where the center spin is influenced by its nearest neighbors: an individual located at the central site adopts the majority sign of the spins in its neighborhood with probability p and the minority sign with probability $q = 1 - p$. This model undergoes a nonequilibrium phase transition at a critical noise $q_c \sim 0.075$, and the critical exponents were found to be the same as those of the equilibrium two-dimensional (2D) Ising model, i.e., $\beta \sim 0.125$, $\gamma \sim 1.75$, and $\nu \sim 1.0$ [15]. After that, some extensions of the original model were proposed. For example, we can cite the interaction between two different classes of agents [16], spins with three-state variables [17,18], and outflow dynamics [19]. The influence of topology was also considered with studies on random [20], hypercubic [21], and hyperbolic lattices [22]; random graphs [23]; small-world networks [24]; and others. Some new universality classes were found for different topologies [17,20–24], but in some of the above-mentioned modifications, the critical exponents are Ising-like, i.e., they are the same as those of the original majority-vote model [16,18,19].

One of the challenges of sociophysics' models is to represent real social systems [14]. A considerable effort was made in this direction with the study of models of opinion dynamics on top of complex networks [25–28] since we know that the topology of real social networks is complex [29–31]. However, as a simpler representation of real social systems, we can also consider the models on regular lattices. In a real community represented by a lattice, people do not always rest on a site of this lattice connected to their nearest neighbors. In addition, a person rarely always has the same friends and/or colleagues (represented by the neighbors). Thus, to study a more realistic model, we can consider a randomly diluted lattice where every site is either empty or carries one individual. The empty sites can be used by the individuals to move around the lattice. Similar studies were performed in other opinion models [32–34], and interesting results were found.

In this work we analyze the critical behavior of the majority-vote model with noise defined on a randomly diluted square lattice for which a site is empty with probability r. We perform Monte Carlo simulations on lattices with different sizes, and our results suggest that the critical noise parameter depends on the lattice dilution. In addition, the critical exponents are r independent, but they are different from those of the original model, defining a new universality class for the majority-vote model.

This work is organized as follows. In Sec. II we present the model and define its microscopic rules. The numerical results as well as the finite-size scaling analysis are discussed in Sec. III, and our conclusions are presented in Sec. IV.

II. MODEL

We have considered the majority-vote model on a random diluted square lattice (size $N = L \times L$) for which a site is empty with probability r, i.e., the mean number of agents on the system is $n = (1 - r)N$. Initially, at every nonempty site i, we assign a random opinion $\sigma_i = \pm 1$ with equal probability, corresponding to the two possible opinions of a certain subject. As in the original model [15], a randomly chosen individual follows the opinion of the majority of its nearest neighbors with probability p and adopts the minority sign of the spins in

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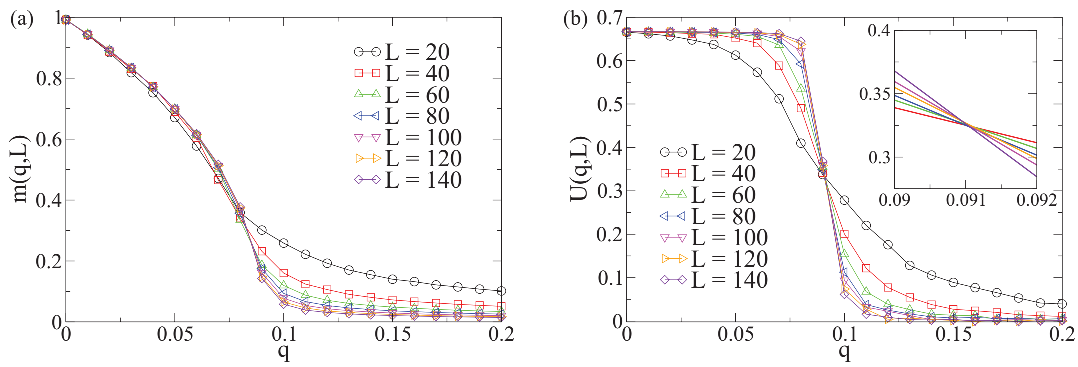

FIG. 1. (Color online) (a) Magnetization per spin m and (b) Binder cumulant U as functions of the noise q for some lattice sizes L for the majority-vote model with lattice dilution $r = 0.3$. Notice the crossing of the Binder cumulant curves near $q_c \sim 0.091$. In order to avoid finite-size effects, we have excluded the smaller size $L = 20$ in the inset.

its neighborhood with probability $q = 1 - p$. In other words, a given spin σ_i is flipped with probability

$$w = \frac{1}{2}\left[1 - (1-2q)\sigma_i S\left(\sum_{\delta=1}^{4}\sigma_{i+\delta}\right)\right], \qquad (1)$$

where $S(x) = \mathrm{sgn}(x)$ if $x \neq 0$ and $S(0) = 0$ and the summation is over the nearest neighbors.

The presence of holes on the lattice is the first modification of the majority-vote model presented in this work. Notice that for holes, $\sigma_{i+\delta} = 0$ in Eq. (1). However, we have studied a second situation in which in addition to the lattice dilution, we have considered that the empty sites on the lattice are used by the individuals to diffuse under a random walk. For the numerical implementation of diffusion in the model, the algorithm is as follows:

(1) we choose a random individual i in the lattice;

(2) the opinion of this agent changes according to the majority rule: we flip the spin at site i with a probability given by Eq. (1) if this individual has at least one active neighbor, i.e., if there is at least one occupied site in the neighborhood of the individual i;

(3) the chosen individual selects randomly a nearest-neighbor site and moves to this site if it is empty.

If we consider no diffusion, we skip the above step (3). We have performed Monte Carlo simulations on lattices with linear sizes $L = 20, 40, 60, 80, 100, 120$, and 140 and periodic boundary conditions for both cases. Time is measured in Monte Carlo steps (MCSs), and one MCS corresponds to N attempts of changing the states of the agents. In other words, we choose randomly an individual as described in the above step (1), and then we follow the rules (2) and (3) [35]. By repeating this procedure N times, we have accomplished one MCS. Concerning the relaxation of the model, we waited between 4×10^4 and 6×10^4 MCSs to reach the steady states with the larger number of MCSs used in the critical region. The time averages were estimated from the next 6×10^4 MCSs after the system reached the steady state. In addition, we have considered 200 independent realizations in the calculation of the configurational averages.

III. NUMERICAL RESULTS

A. Lattice dilution

To study the effects of the lattice dilution on the critical behavior of the model, we have considered the magnetization per spin m, the susceptibility χ, and the Binder cumulant U [36]. For a given realization j of the dynamics, the magnetization

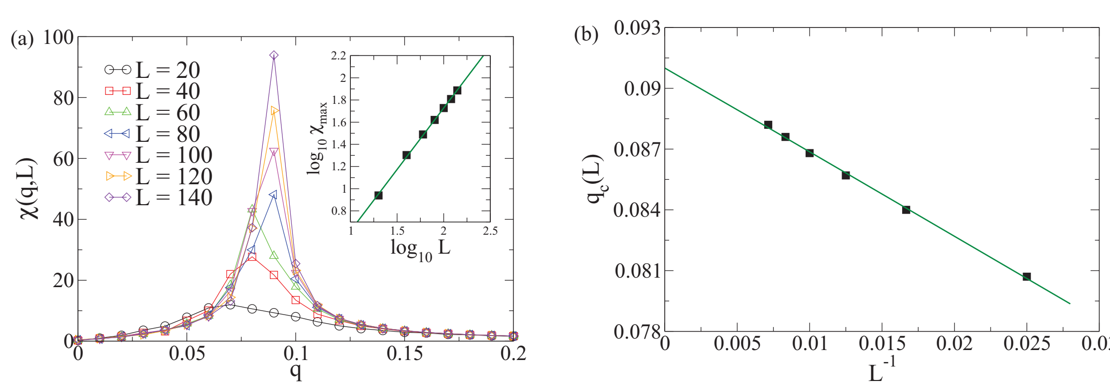

FIG. 2. (Color online) Susceptibility χ as a function of the noise q for some lattice sizes L for the majority-vote model with lattice dilution $r = 0.3$. (a) In the inset we exhibit the log-log plot of $\chi_{\max} = \chi[q_c(L)]$ versus L. Fitting data, we can estimate the critical exponent ratio $\gamma/\nu = 1.089 \pm 0.018$. (b) The pseudocritical noise $q_c(L)$ as a function of L^{-1} is also exhibited. Fitting data, we can obtain a straight line $q_c(L) = 0.091 - 0.417L^{-1}$, giving us $\nu \sim 1.0$ and $q_c = q_c(L \to \infty) \sim 0.091$.

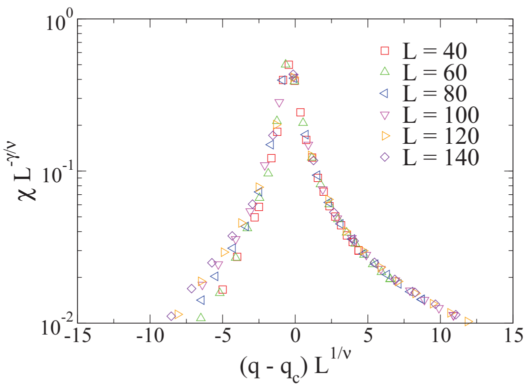

FIG. 3. (Color online) Scaling plot of the susceptibility χ in the log-linear scale, obtained with $q_c = 0.091$, $1/\nu = 1.0$, and $\gamma/\nu = 1.089$. The curves for all values of L fall within a single curve, proving the reliability of the critical exponents. The value of the dilution is $r = 0.3$. We have excluded the smaller size $L = 20$ from this figure in order to minimize the strong finite-size effects for $q < q_c$.

per spin m_j may be obtained from the simulations by

$$\langle m_j \rangle = \left\langle \frac{1}{n_j} \left| \sum_{i=1}^{n_j} \sigma_i \right| \right\rangle, \tag{2}$$

where n_j is the number of active sites of the lattice in realization j and the symbol $\langle \cdots \rangle$ stands for time averages taken in the steady state. In addition to the time average of Eq. (2), we have considered configurational averages, i.e., averages over different realizations. For the magnetization per spin, this average is given by

$$m(q,L) = \frac{1}{s} \sum_{j=1}^{s} \langle m_j \rangle \equiv [\langle m_j \rangle]_s, \tag{3}$$

where s is the number of samples (or realizations) and we have used the notation $[\cdots]_s$ to denote averages over different samples. The other quantities of interest were obtained in a similar way, i.e., we have considered the relation

$$\langle m_j^x \rangle = \left\langle \frac{1}{n_j} \left| \sum_{i=1}^{n_j} \sigma_i \right|^x \right\rangle \tag{4}$$

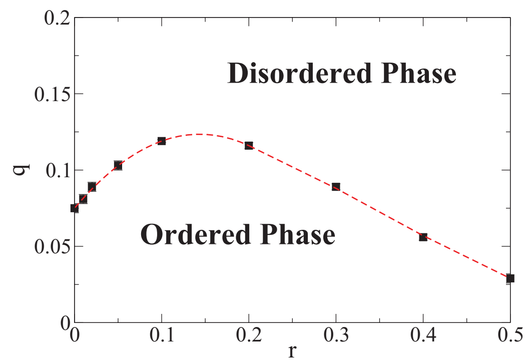

FIG. 5. (Color online) Phase diagram of the majority-vote model with site dilution in the plane q versus r, separating the ordered and the disordered phases. The points are the numerical values $q_c(r)$ estimated from the simulations, and the dashed line is just a guide to the eyes.

for $x = 2$ and $x = 4$. Thus, using Eq. (4), we can obtain the susceptibility χ and the Binder cumulant U from

$$\chi(q,L) = \left[\langle m_j^2 \rangle\right]_s - [\langle m_j \rangle]_s^2, \tag{5}$$

$$U(q,L) = 1 - \frac{\left[\langle m_j^4 \rangle\right]_s}{3\left[\langle m_j^2 \rangle\right]_s^2}. \tag{6}$$

In Fig. 1 we exhibit results for the magnetization per spin [Fig. 1(a)] and the Binder cumulant [Fig. 1(b)] for $r = 0.3$ and some lattice sizes L. We can observe in both graphs a crossing of the curves for different values of L near $q = 0.1$. It is also shown in the inset of Fig. 1(b) a zoom of the Binder cumulant data in the region near the crossing of the curves. In this case, we can estimate the critical noise for $r = 0.3$ ($q_c \sim 0.091$). This value will be confirmed in the following.

In Fig. 2(a) we exhibit the numerical results for the susceptibility χ for $r = 0.3$ and some lattice sizes L, obtained from Eq. (5). Notice that the curves of χ present peaks at certain values $q_c(L)$ [37], which are usually called the pseudocritical noises. In the inset of Fig. 2(a) we show the height of the peaks of the susceptibility as a function of L in the log-log scale. Based on the usual finite-size scaling (FSS)

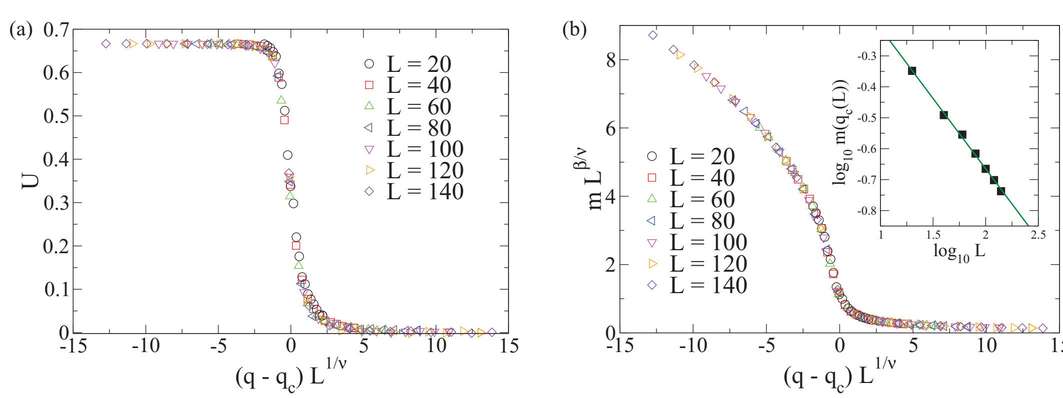

FIG. 4. (Color online) Scaling plot of the Binder cumulant for some lattice sizes L. (a) The best collapse of data gives us $1/\nu = 1.0 \pm 0.03$. (b) Also shown is the collapse of the magnetization per spin. In the inset we exhibit the log-log plot of $m[q_c(L)]$ versus L. Fitting data, we can estimate the critical exponent ratio $\beta/\nu = 0.44 \pm 0.04$. The value of the dilution is $r = 0.3$.

TABLE I. Critical points q_c and the critical ratios γ/ν, $1/\nu$, and β/ν for some values of the dilution probability r. Considering the error bars, the critical ratios do not depend on r.

r	q_c	γ/ν	$1/\nu$	β/ν
0.01	0.081 ± 0.003	1.092 ± 0.014	1.00 ± 0.03	0.41 ± 0.06
0.02	0.089 ± 0.003	1.098 ± 0.011	1.03 ± 0.03	0.40 ± 0.05
0.05	0.103 ± 0.003	1.095 ± 0.012	1.04 ± 0.04	0.40 ± 0.06
0.1	0.119 ± 0.001	1.118 ± 0.015	1.04 ± 0.04	0.40 ± 0.05
0.2	0.116 ± 0.002	1.089 ± 0.018	1.03 ± 0.02	0.42 ± 0.03
0.3	0.091 ± 0.002	1.089 ± 0.015	1.00 ± 0.03	0.44 ± 0.02
0.4	0.056 ± 0.002	1.107 ± 0.013	0.99 ± 0.04	0.45 ± 0.03
0.5	0.029 ± 0.003	1.083 ± 0.021	0.98 ± 0.02	0.45 ± 0.02

equation

$$\chi(q,L) = L^{\gamma/\nu}\tilde{\chi}[(q-q_c)L^{1/\nu}], \tag{7}$$

where $\tilde{\chi}$ is a scaling function, we can fit the data with a straight line, the slope of which gives us the estimate of the critical ratio $\gamma/\nu = 1.089 \pm 0.018$. In addition, based on the FSS equation

$$q_c(L) = q_c - aL^{-1/\nu}, \tag{8}$$

where a is a constant, we can plot the pseudocritical noise $q_c(L)$ as a function of $L^{-1/\nu}$ in order to obtain the critical noise in the thermodynamic limit $L^{-1} \to 0$. This result is exhibited in Fig. 2(b). Observe that the data for $q_c(L)$ as a function of $L^{-1/\nu}$ may be fitted with Eq. (8), giving us $a = 0.417 \pm 0.008$, $1/\nu = 1.0013 \pm 0.0018$, and $q_c = q_c(L \to \infty) = 0.091 \pm 0.002$. This result confirms our first estimate of the critical noise for $r = 0.3$.

Taking into account the above-obtained critical exponents and the critical noise, we exhibit in Fig. 3 the scaling plot of the susceptibility. Observe that the data for χ at different values of L collapse onto a single curve, proving the reliability of the numerical results. The value of the critical exponent $\nu = 1.0$ can be confirmed by the scaling plot of the Binder cumulant data. The result is shown in Fig. 4(a). Based on the FSS equation

$$U(q,L) = \tilde{U}[(q-q_c)L^{1/\nu}], \tag{9}$$

where $\tilde{U}$ is a scaling function, we can estimate $\nu = 1.00 \pm 0.03$ in agreement with the result obtained from Eq. (8). The critical ratio β/ν may be obtained from the magnetization data near q_c, according to the FSS equation

$$m(q,L) = L^{-\beta/\nu}\tilde{m}[(q-q_c)L^{1/\nu}], \tag{10}$$

where $\tilde{m}$ is a scaling function. Thus, plotting the values of $m[q_c(L)]$ versus L in the log-log scale and fitting data, we obtained $\beta/\nu = 0.44 \pm 0.04$ [see the inset of Fig. 4(b)]. Based on the computed critical exponents, we show in Fig. 4(b) the collapse of the magnetization data.

The above procedure was repeated for $r = 0.01, 0.02, 0.05, 0.1, 0.2, 0.4$, and 0.5, and the numerical results for the critical noises q_c and the critical ratios γ/ν, $1/\nu$, and β/ν as functions of r are exhibited in Table I. Observe that q_c depends on r, but the critical exponents do not, considering the error bars. In addition, the scaling relation

$$2\frac{\beta}{\nu} + \frac{\gamma}{\nu} = d, \tag{11}$$

where $d = 2$ is the dimension of the lattice, is satisfied for all values of r within the error bars. Observe also that the magnetic critical exponents are different from those of the original model for which the exponents are given by $\gamma/\nu \sim 1.75$, $1/\nu \sim 1.0$, and $\beta/\nu \sim 0.125$ [15]. The thermal exponent ν remains the same. Thus, the lattice dilution plays an important role in the model, modifying the critical behavior of the system, which reflects on different magnetic critical exponents. Nonetheless, our results indicate a universality of the critical exponents along the order-disorder frontier (for $r > 0$), suggesting that we have a new universality class for the majority-vote model with noise.

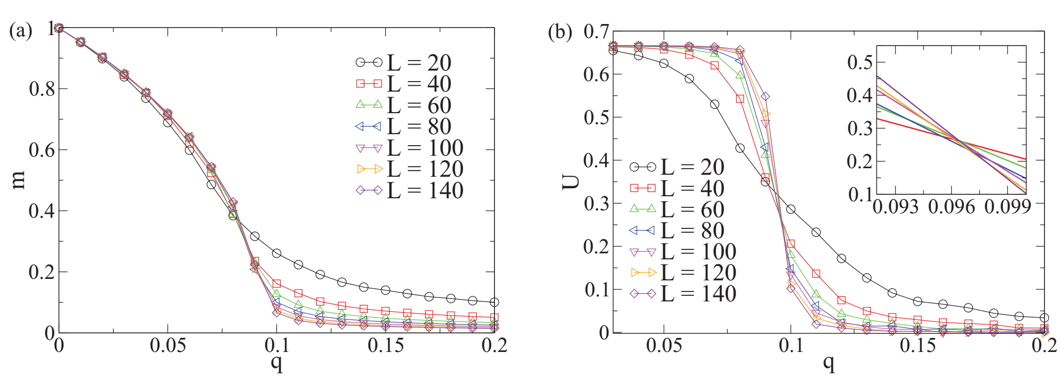

FIG. 6. (Color online) (a) Magnetization per spin m and (b) Binder cumulant U as functions of the noise q for some lattice sizes L for the majority-vote model with lattice dilution $r = 0.3$ and agents' diffusion. Notice the crossing of the Binder cumulant curves near $q_c \sim 0.096$. In order to avoid finite-size effects, we have excluded the smaller size $L = 20$ in the inset.

TABLE II. Critical points q_c and the critical ratios γ/ν, $1/\nu$, and β/ν for some values of the dilution probability r for the model with agents' diffusion. Considering the error bars, the critical ratios do not depend on r.

r	q_c	γ/ν	$1/\nu$	β/ν
0.01	0.090 ± 0.004	1.113 ± 0.019	1.02 ± 0.03	0.39 ± 0.05
0.02	0.095 ± 0.003	1.109 ± 0.012	1.03 ± 0.03	0.40 ± 0.04
0.05	0.113 ± 0.004	1.107 ± 0.014	1.02 ± 0.03	0.43 ± 0.04
0.1	0.129 ± 0.003	1.081 ± 0.014	1.03 ± 0.02	0.43 ± 0.03
0.2	0.125 ± 0.002	1.094 ± 0.013	1.00 ± 0.02	0.45 ± 0.03
0.3	0.096 ± 0.003	1.092 ± 0.012	1.00 ± 0.03	0.44 ± 0.04
0.4	0.063 ± 0.002	1.098 ± 0.016	0.98 ± 0.02	0.46 ± 0.03
0.5	0.035 ± 0.003	1.083 ± 0.015	1.00 ± 0.02	0.45 ± 0.02

Taking into account the critical noises q_c obtained from the simulations for different values of the lattice dilution r, we exhibit in Fig. 5 the phase diagram of the model in the plane q versus r. The squares are the critical values q_c estimated from the Monte Carlo simulations whereas the dashed line is just a guide to the eyes. These points separate an ordered phase for which the system presents ferromagnetic ordering, from a disordered phase, where the system is in a paramagnetic state. Notice that the exponents along this phase boundary remain the same except for $r = 0$, suggesting a universality of the model under variations on the lattice dilution r.

B. Lattice dilution and agents' diffusion

To study the effects of the agents' mobility on the critical behavior of the model, we have considered the same quantities of the last section, i.e., the magnetization per spin m, the susceptibility χ, and the Binder cumulant U. The results for these quantities are qualitatively similar to the ones obtained for the model with no diffusion. As an example, we exhibit in Fig. 6 data for the magnetization per spin and the Binder cumulant for $r = 0.3$. Notice that the value of the critical noise in this case, $q_c(r = 0.3) \sim 0.096$, is greater than the one obtained in the last section. Nonetheless, we have performed a FSS analysis of the numerical data, and we have obtained the same critical exponents in both cases, considering the error bars. The same results were found for other values of the probability r as exhibited in Table II.

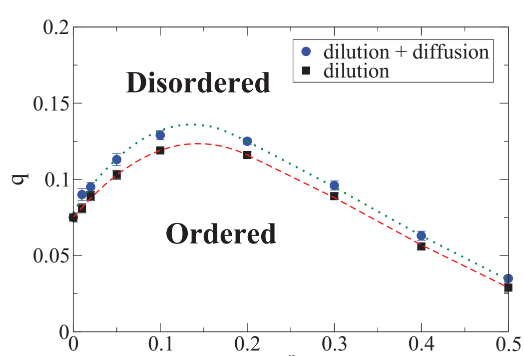

FIG. 7. (Color online) Comparative phase diagram of the majority-vote model in the plane q versus r, separating the ordered and the disordered phases. The squares are the numerical estimates for the critical values $q_c(r)$ for the model with site dilution whereas the circles are the critical points $q_c(r)$ for the case with site dilution and agents' diffusion. The lines are just guides for the eye.

Thus, the diffusion of agents does not modify the critical behavior of the model: the only difference is a shift of the critical noise. In particular, we have $q_c^{\text{diffusion}} > q_c^{\text{no diffusion}}$. In other words, the dilution of sites on a square lattice provokes the appearance of a new universality class for the majority-vote model with noise, which is robust with respect to the agents' diffusion. A comparative phase diagram of the two formulations of the model is shown in Fig. 7.

IV. FINAL REMARKS

In this work we have studied a modified version of the majority-vote model with noise. We have considered the model on a diluted square lattice on which a site is empty with a probability r. The critical behavior of the model was analyzed by means of Monte Carlo simulations on lattices with linear sizes up to $L = 140$. We have verified that the critical noises q_c depend on the dilution r, but the critical exponents β, γ, and ν do not, the magnetic exponents β and γ presenting different values in comparison with the original model [15], while the thermal exponent ν remains the same. In addition, if we allow the agents to diffuse through the lattice, the exponents are not modified. These results suggest that we have a new universality class for the model, and based on the numerical data, we may conjecture that the values of the exponents in this universality class are $\beta \sim 0.45$, $\gamma \sim 1.1$, and $\nu \sim 1.0$, which satisfy the scaling relation $2\beta + \gamma = d\nu = 2$.

It is important to observe that the standard majority-vote model with noise falls in the same universality class of the equilibrium 2D Ising model [15]. In addition, the inclusion of site dilution in the 2D Ising model does not affect the exponents, i.e., the site-diluted and the pure Ising models are in the same universality class [38]. However, the same behavior is not observed in the majority-vote model: our simulations suggest that the site dilution alters the universality class of the model.

ACKNOWLEDGMENTS

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[1] S. Galam, Int. J. Mod. Phys. C **19**, 409 (2008).
[2] K. Sznajd-Weron and J. Sznajd, Int. J. Mod. Phys. C **11**, 1157 (2000).
[3] D. Stauffer, A. O. Sousa, and S. Moss de Oliveira, Int. J. Mod. Phys. C **11**, 1239 (2000).
[4] N. Crokidakis and F. L. Forgerini, Phys. Lett. A **374**, 3380 (2010).
[5] N. Crokidakis and P. M. C. de Oliveira, J. Stat. Mech. (2011) P11004.
[6] N. Crokidakis, Phys. A (Amsterdam, Neth.) **391**, 1729 (2012).
[7] N. Crokidakis and F. L. Forgerini, Braz. J. Phys. **42**, 125 (2012).
[8] M. Lallouache, A. S. Chakrabarti, A. Chakraborti, and B. K. Chakrabarti, Phys. Rev. E **82**, 056112 (2010).
[9] P. Clifford and A. Sudbury, Biometrika **60**, 581 (1973).
[10] R. Holley and T. Liggett, Ann. Probab. **3**, 643 (1975).
[11] M. Lewenstein, A. Nowak, and B. Latané, Phys. Rev. A **45**, 763 (1992).
[12] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, Adv. Complex Syst. **3**, 87 (2000).
[13] R. Hegselmann and U. Krause, J. Artif. Soc. Soc. Simul. **5**, 2 (2002).
[14] C. Castellano, S. Fortunato, and V. Loreto, Rev. Mod. Phys. **81**, 591 (2009).
[15] M. J. de Oliveira, J. Stat. Phys. **66**, 273 (1992).
[16] A. L. M. Vilela and F. G. B. Moreira, Phys. A (Amsterdam, Neth.) **388**, 4171 (2009).
[17] D. F. F. Melo, L. F. C. Pereira, and F. G. B. Moreira, J. Stat. Mech. (2010) P11032.
[18] F. W. S. Lima, Phys. A (Amsterdam, Neth.) **391**, 1753 (2012).
[19] C. I. N. Sampaio-Filho and F. G. B. Moreira, Phys. Rev. E **84**, 051133 (2011).
[20] F. W. S. Lima, U. L. Fulco, and R. N. Costa Filho, Phys. Rev. E **71**, 036105 (2005).
[21] J.-S. Yang, I.-M. Kim, and W. Kwak, Phys. Rev. E **77**, 051122 (2008).
[22] Z.-X. Wu and P. Holme, Phys. Rev. E **81**, 011133 (2010).
[23] L. F. C. Pereira and F. G. Brady Moreira, Phys. Rev. E **71**, 016123 (2005).
[24] P. R. A. Campos, V. M. de Oliveira, and F. G. Brady Moreira, Phys. Rev. E **67**, 026104 (2003).
[25] F. Wu, B. A. Huberman, L. A. Adamic, and J. R. Tyler, Phys. A (Amsterdam, Neth.) **337**, 327 (2004).
[26] K. Suchecki, V. M. Eguíluz, and M. S. Miguel, Europhys. Lett. **69**, 228 (2005).
[27] A. T. Bernardes, U. M. S. Costa, A. D. Araujo, and D. Stauffer, Int. J. Mod. Phys. C **12**, 159 (2001).
[28] J. O. Indekeu, Phys. A (Amsterdam, Neth.) **333**, 461 (2004).
[29] J. Scott, *Social Network Analysis: A Handbook* (Sage Publications, London, 2000).
[30] F. Liljeros, C. R. Edling, L. A. N. Amaral, H. E. Stanley, and Y. Aberg, Nature (London) **411**, 907 (2001).
[31] A.-L. Barabási, H. Jeong, Z. Neda, E. Ravasz, A. Schubert, and T. Vicsek, Phys. A (Amsterdam, Neth.) **311**, 590 (2002).
[32] A. O. Sousa, T. Yu-Song, and M. Ausloos, Eur. Phys. J. B **66**, 115 (2008).
[33] Q. Guo, J.-G. Liu, B.-H. Wang, T. Zhou, X.-W. Chen, and Y.-H. Yao, Chin. Phys. Lett. **25**, 773 (2008).
[34] S. Galam, Eur. Phys. J. B **25**, 403 (2002).
[35] Step (3) is considered only if we take into account the agents' diffusion.
[36] K. Binder, Z. Phys. B **43**, 119 (1981).
[37] We performed simulations near the values corresponding to the maxima of χ in Fig. 2(a) for each value of L in order to obtain a good estimate for $q_c(L)$ and $\chi_{\max} = \chi[q_c(L)]$.
[38] R. B. Stinchcombe, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1983), Vol. 7.