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Generalized space and linear momentum operators in quantum mechanics

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We propose a modification of a recently introduced generalized translation operator, by including a q-exponential factor, which implies in the definition of a Hermitian deformed linear momentum operator $\hat{p}_q$, and its canonically conjugate deformed position operator $\hat{x}_q$. A canonical transformation leads the Hamiltonian of a position-dependent mass particle to another Hamiltonian of a particle with constant mass in a conservative force field of a deformed phase space. The equation of motion for the classical phase space may be expressed in terms of the generalized dual q-derivative. A position-dependent mass confined in an infinite square potential well is shown as an instance. Uncertainty and correspondence principles are analyzed. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4884299]

INTRODUCTION

Systems consisting of particles with position-dependent mass have been discussed by several researchers since few past decades. Applications of such systems may be found in semiconductor theory,[1] ^{4}He impurity in homogeneous liquid ^{3}He,[2] nonlinear optics,[3] studies of inversion potential for NH_3 in density functional theory,[4] particle physics,[5] and astrophysics.[6]

Recently, Costa Filho *et al.*[7,8] introduced a generalized translation operator which produces infinitesimal displacements related to the q-algebra,[9,10] i.e.,

$$\hat{T}_\gamma(\varepsilon)|x\rangle \equiv |x + \varepsilon + \gamma x \varepsilon\rangle, \tag{1}$$

where γ is a parameter with dimension of inverse length. This operator leads to a generator operator of spatial translations corresponding to a position-dependent linear momentum given by $\hat{p}_\gamma = (\hat{1} + \gamma \hat{x})\hat{p}$, and consequently a particle with position-dependent mass. It has been used to solve problems of particles with position-dependent mass in the quantum formalism,[11,12] and more recently, it has also been applied to analyze systems with position-dependent effective mass in nanostructures.[13,14] More specifically, quantum confinement in Si and Ge nanostructures were experimentally investigated, and the theoretical treatment with the deformed translation operator has lead to a change in the effective mass and an increased confinement energy. The authors have found a relation between the parameter γ and the nanostructure diameter. This linear momentum operator is not Hermitian, which led Mazharimousavi[15] to introduce a modification in its definition. Other generalizations have also appeared in the literature, particularly a nonlinear version of Schrödinger, Klein-Gordon, and Dirac equations.[16–19]

The paper is organized as follows: we first present a brief review of some properties of the q-algebra. Next, we propose a modification of the translation operator, and an alternative deduction for the generator of generalized infinitesimal translations. We also present a generalized space

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operator in which is valid the canonical commutation relation with the generalized linear momentum operator already introduced. These operators constitute a canonical transformation which maps a particle with constant mass into another one with position-dependent mass. We analyze the classical analogues of these operators and compare the results of classical and quantum formalisms through the correspondence principle. Specifically, the problem of a particle with position-dependent mass confined in an infinite square well is considered, as well as the uncertainty principle for this problem.

DEFORMED FUNCTIONS AND DERIVATIVES

The q-exponential is a deformation of the ordinary exponential function, and it plays a central role in nonextensive statistical mechanics, defined by

$$\exp_q x \equiv [1+(1-q)x]_+^{1/(1-q)}, \tag{2}$$

where the dimensionless parameter q controls the generalization of the exponential function, and $[A]_+ \equiv \max\{A, 0\}$.[20] The q-exponential function satisfies $\exp_q(a)\exp_q(b) = \exp_q(a \oplus_q b)$ and $\exp_q(a)/\exp_q(b) = \exp_q(a \ominus_q b)$, with $\oplus_q$ representing the q-addition operator defined by $a \oplus_q b \equiv a + b + (1-q)ab$, and $\ominus_q$ the q-subtraction, $a \ominus_q b \equiv \frac{a-b}{1+(1-q)b}$, with $b \neq 1/(q-1)$ (a and b shall be dimensionless quantities).[9,10] The q-logarithm function is the inverse of q-exponential function, and it is given by

$$\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \quad (x > 0). \tag{3}$$

It is possible to define a generalization of the derivative operator, based on these deformed algebraic operators.[10] Particularly,

$$\begin{aligned} D_q f(u) &\equiv \lim_{u' \to u} \frac{f(u') - f(u)}{u' \ominus_q u} \\ &= \lim_{\Delta u \to 0} \frac{f(u \oplus_q \Delta u) - f(u)}{\Delta u} \\ &= [1+(1-q)u]\frac{df(u)}{du}, \end{aligned} \tag{4}$$

of which the q-exponential is an eigenfunction. There is a dual q-derivative,

$$\begin{aligned} \widetilde{D}_q f(u) &\equiv \lim_{u' \to u} \frac{f(u') \ominus_q f(u)}{u' - u} \\ &= \frac{1}{1+(1-q)f(u)}\frac{df(u)}{du}, \end{aligned} \tag{5}$$

that satisfies $\widetilde{D}_q \ln_q u = 1/u$. These operators obey $D_q y(x) = [\widetilde{D}_q x(y)]^{-1}, \ \forall q$. The ordinary derivative is recovered for $q = 1$ in both cases (ordinary derivative is self-dual), as well as $dy/dx = (dx/dy)^{-1}$.

The deformed derivative operator $D_q f(u)$ may be understood as the rate of variation of the function $f(u)$ with respect to a nonlinear variation of the independent variable u, $u \oplus_q \Delta u = u + \Delta u + (1-q)u\Delta u$. Similarly, the deformed derivative operator $\widetilde{D}_q f(u)$ may be viewed as the rate of a nonlinear variation of the function $f(u)$ with respect to the ordinary variation of the independent variable u.

Furthermore, considering a real variable u, we have

$$d_q u \equiv \lim_{u' \to u} u' \ominus_q u = \frac{du}{1+(1-q)u} = du_q \tag{6}$$

with

$$u_q \equiv \ln[1+(1-q)u]/(1-q) = \ln(\exp_q u). \tag{7}$$

We shall call u_q a deformed q-number, for brevity. This deformed number has already been defined in Ref. 21. These relations express an interesting feature: the (dual) q-derivative of the ordinary dependent variable f with respect to the ordinary independent variable u is equal to the ordinary derivative of the ordinary (deformed) dependent variable f (f_q) with respect to the deformed (ordinary) independent variable u_q (u), i.e., $D_q f(u) = df(u)/d_q u = df(u)/du_q$ ($\widetilde{D}_q f(u) = d_q f(u)/du = df_q(u)/du$). Let $f(u) = u$, we get the curious relation $D_q u = [\widetilde{D}_q u]^{-1} = 1 + (1-q)u$.

GENERALIZED TRANSLATION, LINEAR MOMENTUM, AND SPACE OPERATORS

We introduce a non-normalized generalized phase factor in Eq. (1) as

$$\hat{T}_q(\varepsilon)|x\rangle \equiv \exp_q\left[\frac{ig(x)\varepsilon}{\hbar}\right]\left|x+\varepsilon+\frac{1-q}{\xi}x\varepsilon\right\rangle = \exp_q\left[\frac{ig(x)\varepsilon}{\hbar}\right]\left|\xi(\tilde{x}\oplus_q\tilde{\varepsilon})\right\rangle, \tag{8}$$

where $g(x)$ is a continuous function with dimension of linear momentum ($g(x) = 0$ recovers Eq. (1)), ε is an infinitesimal displacement, ξ is a characteristic length, and $\tilde{x} \equiv x/\xi$ is the dimensionless position. The symbol γ in Eq. (1) (as it appears in Ref. 7) has been here replaced with $\gamma_q \equiv (1-q)/\xi$, once the q-addition shall be used with dimensionless variables. The usual case is recovered for $q \to 1$ ($\gamma_q \to 0$).

The q-exponential of an imaginary number yields generalized trigonometric functions[21] that can be written as $\exp_q(\pm ix) = \rho_q(x)\exp_1(\pm ix)$, with $\rho_q^2(x) = \exp_q(ix)\exp_q(-ix) = \exp_q[(1-q)x^2]$ ($x \in \mathbb{R}$); $\rho_q(x)$ is the norm of the q-exponential. $q = 1$ recovers the usual exponential function, and the q-exponential factor reduces to a usual phase factor with unitary norm.

Similar to the operator defined by Eq. (1), $\hat{T}_q(\varepsilon)$ also forms a group, i.e.,

$$\hat{T}_q(\xi d\tilde{x}_1)\hat{T}_q(\xi d\tilde{x}_2)|0\rangle = \hat{T}_q(\xi(d\tilde{x}_1 \oplus_q d\tilde{x}_2))|0\rangle. \tag{9}$$

Application of the operator $\hat{T}_q(\varepsilon)$ on state $|0\rangle$, repeated n times, leads to

$$\hat{T}_q^n(\varepsilon)|0\rangle = \exp_q\left[n \odot_q \frac{ig(x)\varepsilon}{\hbar}\right]|n \odot_q \varepsilon\rangle, \tag{10}$$

where $n \odot_q x$ is a generalized product:[10]

$$n \odot_q x = \frac{1}{1-q}\Big\{\big[1+(1-q)x\big]^n - 1\Big\}. \tag{11}$$

(Not to confound the generalized product $n \odot_q x$ with another generalization, frequently known as q-product, $a \otimes_q b \equiv \mathrm{sign}(a)\mathrm{sign}(b)[|a|^{1-q} + |b|^{1-q} - 1]_+^{1/(1-q)}$.[9,10]) This expression may be analytically extended for $n \in \mathbb{R}$. Particularly, for $x = 1$, $n \odot_q 1$ is identified with the Heine deformed number (see Refs. 22 and also 23 for a possible connection with quantum groups).

Let $|\psi_\varepsilon\rangle \equiv \hat{T}_q(\varepsilon)|\psi\rangle$. The effect of the operator $\hat{T}_q(\varepsilon)$ on state $|\psi\rangle$ is

$$|\psi_\varepsilon\rangle = \hat{T}_q(\varepsilon)\int |x\rangle\langle x|\psi\rangle dx = \int \exp_q\left[\frac{ig(x)\varepsilon}{\hbar}\right]|\xi(\tilde{x}\oplus_q\tilde{\varepsilon})\rangle\langle x|\psi\rangle dx. \tag{12}$$

If $\psi_\varepsilon(x) \equiv \langle x|\hat{T}_q(\varepsilon)|\psi\rangle$, we have

$$\psi_\varepsilon(x) = \exp_q\left[\frac{i\varepsilon}{\hbar}g\left(\xi(\tilde{x}\ominus_q\tilde{\varepsilon})\right)\right]\frac{\psi(\xi(\tilde{x}\ominus_q\tilde{\varepsilon}))}{1+\gamma_q\varepsilon}, \tag{13}$$

where $\psi(x) = \langle x|\psi\rangle$.

We set forward a simple application: let $|\psi\rangle$ be a normalized Gaussian state with width σ, and centered at $x = 0$, in the coordinate basis $\{|x\rangle\}$. The effect of $\hat{T}_q(\varepsilon)$ on $|\psi\rangle$ leads to

$$\psi_\varepsilon(x) = \frac{e^{-(x-\varepsilon)^2/2\sigma_q^2}}{\sigma_q\sqrt{2\pi}} \exp_q\left[\frac{i\varepsilon}{\hbar} g\left(\frac{x-\varepsilon}{1+\gamma_q\varepsilon}\right)\right], \tag{14}$$

with $\sigma_q = \sigma(1 + \gamma_q\varepsilon)$, which is normalized up to $\mathcal{O}(\varepsilon)$. If $g(x) = 0$, the effect of $\hat{T}_q(\varepsilon)$ on a Gaussian packet yields a shift in x and an increase in its width for $1 + \gamma_q\varepsilon > 0$, or a decrease for $1 + \gamma_q\varepsilon < 0$.

Let the expected value $\langle\hat{x}\rangle_\varepsilon = \int \psi_\varepsilon^*(x')x'\psi_\varepsilon(x')dx'$, by changing the variable of integration $x = \xi(\tilde{x}' \ominus_q \tilde{\varepsilon})$, we have

$$\langle\hat{x}\rangle_\varepsilon = \int dx\,\psi^*(x)(x + \varepsilon + \gamma_q x\varepsilon)\psi(x)\frac{e_q^{(1-q)\varepsilon^2 g^2(x)/\hbar^2}}{1+\gamma_q\varepsilon}. \tag{15}$$

The first order approximation in ε is represented by the q-addition

$$\langle\hat{x}\rangle_\varepsilon = \langle\hat{x}\rangle + \varepsilon + \gamma_q\langle\hat{x}\rangle\varepsilon. \tag{16}$$

Equation (9) naturally suggests the definition

$$\hat{T}_q(\varepsilon) \equiv \exp_q\left(-\frac{i\varepsilon\hat{p}_q}{\hbar}\right), \tag{17}$$

$\hat{p}_q$ is the generator of generalized infinitesimal translations. Expanding $\hat{T}_q(\varepsilon)$, and $\psi_\varepsilon(x)$ (Eq. (13)), up to the first order in ε, we get

$$\hat{T}_q(\varepsilon) = \hat{1} - \frac{i\varepsilon\hat{p}_q}{\hbar} + \ldots, \tag{18}$$

$$\begin{aligned}\psi_\varepsilon(x) = {} & (1 - \gamma_q\varepsilon + \ldots)(1 + \varepsilon A + \ldots .) \\ & \times\left[\psi(x) - \varepsilon(1+\gamma_q x)\frac{d\psi}{dx} + \ldots\right],\end{aligned} \tag{19}$$

where A is a constant taken from the expansion of $\exp_q(ig(x)\varepsilon/\hbar)$ in powers of ε, and we have

$$\langle x|\hat{p}_q|\psi\rangle = -i\hbar\frac{d}{dx}[(1+\gamma_q x)\psi(x)] + i\hbar A\psi(x). \tag{20}$$

Imposing $\hat{p}_q$ as Hermitian implies $A = \gamma_q/2$, thus

$$\hat{p}_q = \hat{p}(\hat{1} + \gamma_q\hat{x}) + \frac{1}{2}i\hbar\gamma_q\hat{1} = (\hat{1} + \gamma_q\hat{x})\hat{p} - \frac{1}{2}i\hbar\gamma_q\hat{1}, \tag{21}$$

i.e.,

$$\hat{p}_q = \frac{(\hat{1} + \gamma_q\hat{x})\hat{p}}{2} + \frac{\hat{p}(\hat{1} + \gamma_q\hat{x})}{2}, \tag{22}$$

with $[\hat{x}, \hat{p}] = i\hbar\hat{1}$.

We introduce a generalized space operator $\hat{x}_q$ such that $[\hat{x}_q, \hat{p}_q] = i\hbar\hat{1}$. Recalling the property $[f(\hat{x}), \hat{p}] = i\hbar f'(\hat{x})$, with $\hat{x}_q = f(\hat{x})$, we arrive at

$$\hat{x}_q = \frac{\ln(\hat{1} + \gamma_q\hat{x})}{\gamma_q} = \xi\ln[\exp_q(\hat{x}/\xi)]. \tag{23}$$

The transformation (23) has already appeared in a different context, as the real part of a transformation of a complex number z into a kind of generalized complex number $\zeta_q = \ln\exp_q z$, which allows expressing the q-Euler formula as $\exp_q z = \exp_1 \zeta_q$.[21] Even before that, the transformation (23) had also appeared connecting Tsallis (nonadditive) entropy with Rényi (additive) entropy.[24]

CLASSICAL ANALOG FOR DEFORMED OPERATORS

Equations (22) and (23), that naturally emerge from a nonlinear space, form a canonical transformation that maps the Hamiltonian $\hat{K}(\hat{x}_q, \hat{p}_q)$ of a particle with constant mass into another one, $\hat{H}(\hat{x}, \hat{p})$, with a particle of position-dependent mass.

According to Ehrenfest's theorem, the time evolution of the expectation values of the space $\hat{x}$ and linear momentum $\hat{p}$ operators are given, respectively, by

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle(\hat{1}+\gamma_q\hat{x})^2\hat{p}\rangle}{2m} + \frac{\langle\hat{p}(\hat{1}+\gamma_q\hat{x})^2\rangle}{2m} \tag{24a}$$

and

$$\frac{d\langle\hat{p}\rangle}{dt} = -\frac{\gamma_q\langle(\hat{1}+\gamma_q\hat{x})\hat{p}^2\rangle}{2m} - \frac{\gamma_q\langle\hat{p}^2(\hat{1}+\gamma_q\hat{x})\rangle}{2m} - \left\langle\frac{dV}{d\hat{x}}\right\rangle, \tag{24b}$$

where we have used the following commutation relations:

$$[\hat{x}, \hat{p}_q^2] = i\hbar(\hat{1}+\gamma_q\hat{x})^2\hat{p} + i\hbar\hat{p}(\hat{1}+\gamma_q\hat{x})^2 \tag{25}$$

and

$$[\hat{p}, \hat{p}_q^2] = -i\hbar\gamma_q(\hat{1}+\gamma_q\hat{x})\hat{p}^2 - i\hbar\gamma_q\hat{p}^2(\hat{1}+\gamma_q\hat{x}). \tag{26}$$

The operators $\hat{x}_q$ and $\hat{p}_q$ present the following classical analogs:

$$p_q = (1+\gamma_q x)p \tag{27a}$$

and

$$x_q = \frac{\ln(1+\gamma_q x)}{\gamma_q} = \xi\ln\left[\exp_q(x/\xi)\right], \tag{27b}$$

with $\{x_q, p_q\}_{(x,p)} = 1$. The generating function of the canonical transformations given by Eqs. (27) is $\Phi(x_q, p) = -p(e^{\gamma_q x_q} - 1)/\gamma_q$.

As an application, let us address a constant mass particle and linear momentum p_q under the influence of a conservative force with potential $V(x_q)$, whose Hamiltonian is

$$K(x_q, p_q) = \frac{p_q^2}{2m} + V(x_q). \tag{28}$$

The canonical transformations (27) lead to the new Hamiltonian (see, for instance, Ref. 25),

$$H(x, p) = \frac{p^2}{2m(x)} + V(x), \tag{29}$$

where the particle mass depends on the position x as

$$m(x) = \frac{m}{(1+\gamma_q x)^2}. \tag{30}$$

The equation of motion is

$$\dot{p} = -\frac{\gamma_q(1+\gamma_q x)p^2}{m} - \frac{dV(x)}{dx}, \tag{31}$$

with $p = m(x)\dot{x}$, thus

$$m\left[\frac{\ddot{x}}{(1+\gamma_q x)^2} - \frac{\gamma_q\dot{x}^2}{(1+\gamma_q x)^3}\right] = -\frac{dV(x)}{dx}. \tag{32}$$

This equation may be conveniently rewritten as

$$m\widetilde{D}^2_{\gamma_q,t}x(t) = F(x), \tag{33}$$

i.e., a deformed Newton's law for a space with nonlinear displacements, where $\widetilde{D}_{q,u}f(u)$ is the dual q-derivative defined by Eq. (5). The second q-derivative must be taken as

$$\widetilde{D}^2_{q,u}f(u)=\frac{1}{1+(1-q)f(u)}\frac{d}{du}\left[\frac{1}{1+(1-q)f(u)}\frac{df}{du}\right], \tag{34}$$

similar to what was done in the (different) generalized derivative introduced by Ref. 16. (That generalized derivative is defined as $\mathcal{D}_q f(u)=[f(u)]^{1-q}df(u)/du$, and, for the particular case $f(u)=\exp_q(u)$, it coincides with Eq. (4).)

The generalized displacement of a position-dependent mass in a usual space ($d_q x$) is mapped into a constant mass in a deformed space with usual displacement (dx_q): $d_q x\equiv\xi\left[\left(\frac{x+dx}{\xi}\right)\ominus_q\left(\frac{x}{\xi}\right)\right]=\frac{dx}{1+\gamma_q x}\equiv dx_q$. The temporal evolution is governed by the generalized dual derivative, $\widetilde{D}_{\gamma_q,t}x=\frac{1}{1+\gamma_q x}\frac{dx}{dt}$.

The probability $P_{\text{classic}}dx\propto dx/v$ to find a classical particle with position-dependent mass given by Eq. (30), between x and $x+dx$, constrained to $0\le x\le L$, and free of forces, is

$$P_{\text{classic}}dx=\frac{\gamma_q}{(1+\gamma_q x)\ln(1+\gamma_q L)}dx. \tag{35}$$

Note that the probability density P_{classic} is independent of the initial condition, and the uniform distribution $P_{\text{classic}}\to 1/L$ is recovered in the limit $\gamma_q\to 0$.

The first and second moments of both position and momentum of the classical distribution are

$$\overline{x}=\frac{\gamma_q L-\ln(1+\gamma_q L)}{\gamma_q\ln(1+\gamma_q L)}, \tag{36a}$$

$$\overline{x^2}=\frac{\gamma_q^2L^2-2\gamma_q L+2\ln(1+\gamma_q L)}{2\gamma_q^2\ln(1+\gamma_q L)}, \tag{36b}$$

and

$$\overline{p}=0, \tag{36c}$$

$$\overline{p^2}=2mE\frac{[(1+\gamma_q L)^2-1]}{2(1+\gamma_q L)^2\ln(1+\gamma_q L)}, \tag{36d}$$

where $\lim_{\gamma_q\to 0}\overline{x}=L/2$, $\lim_{\gamma_q\to 0}\overline{x^2}=L^2/3$, $\lim_{\gamma_q\to 0}\overline{p^2}=2mE$, and E is the energy of the particle.

PARTICLE CONFINED IN AN INFINITE SQUARE WELL

Consider a system described by the Hamiltonian operator $\hat{K}$ in the coordinate basis $\{|\hat{x}_q\rangle\}$. The time independent Schrödinger equation for the free particle in the basis $\{|\hat{x}_q\rangle\}$ is

$$\frac{1}{2m}\hat{p}_q^2|\psi\rangle=E|\psi\rangle. \tag{37}$$

Using Eq. (22), we have

$$-\frac{\hbar^2(1+\gamma_q x)^2}{2m}\frac{d^2\psi}{dx^2}-\frac{\hbar^2\gamma_q(1+\gamma_q x)}{m}\frac{d\psi}{dx}-\frac{\hbar^2\gamma_q^2}{8m}\psi(x)=E\psi(x), \tag{38}$$

which can be rewritten in the form

$$u^2\frac{d^2\psi(u)}{du^2}+au\frac{d\psi(u)}{du}+b\psi(u)=0, \tag{39}$$

with $u(x)=1+\gamma_q x$, $a=2$, and $b=\frac{2m}{\hbar^2}\left(E+\frac{\hbar^2\gamma_q^2}{8m}\right)$.

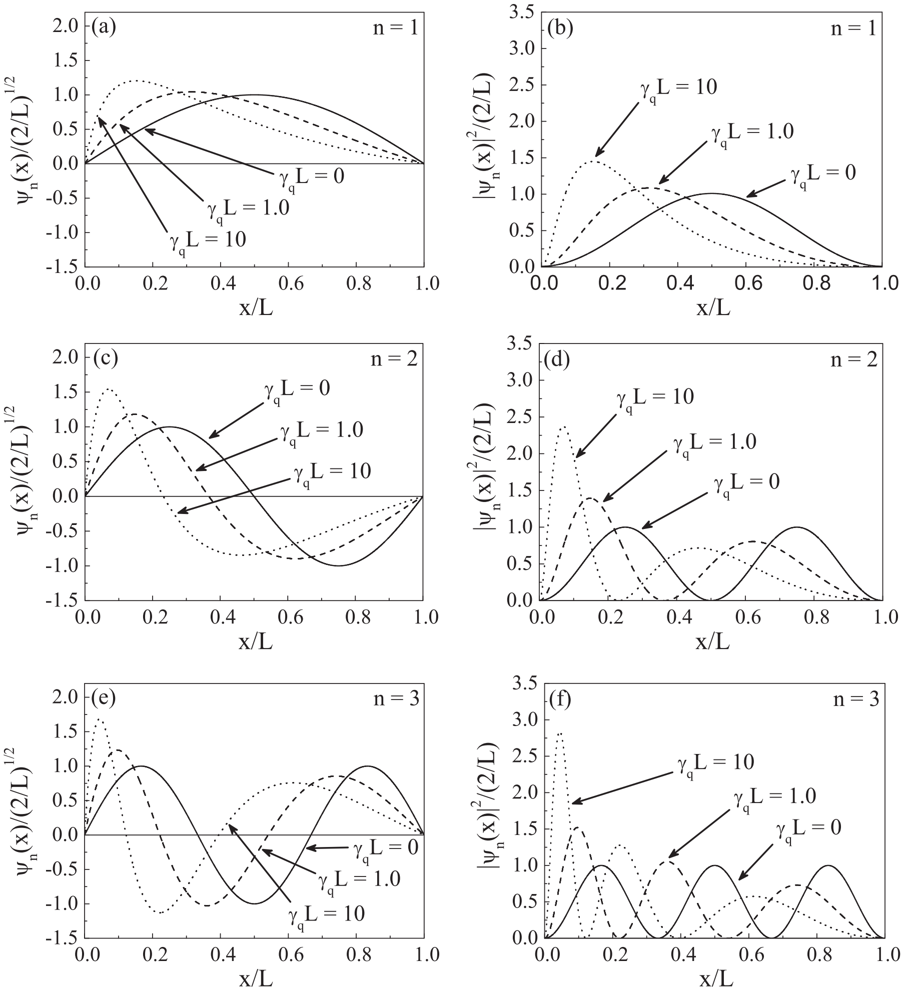

FIG. 1. Wave functions $\psi_n(x)$ (left column) and probability densities $|\psi_n(x)|^2$ (right column), conveniently scaled, for a particle confined in an infinite square well within a generalized space for different values of $\gamma_q L$ (indicated, usual case $\gamma_q L = 0$ is shown, for comparison). (a) and (b): $n = 1$ (ground state), (c) and (d): $n = 2$ (first exited state), (d) and (f): $n = 3$ (second exited state).

Similar to what was done in Refs. 7 and 15, Eq. (38) corresponds to a position-dependent mass particle according to Eq. (30). The solution of Eq. (38) is given by

$$\psi(x) = \frac{\psi_0}{\sqrt{1+\gamma_q x}} \exp\left[\pm\frac{ik}{\gamma_q}\ln(1+\gamma_q x)\right] = \frac{\psi_0}{\sqrt{1+(1-q)x/\xi}}\left[\exp_q(x/\xi)\right]^{\pm ik\xi} \tag{40}$$

and presents a singularity at $x = -1/\gamma_q$.

For a particle inside, an infinite square potential well between $x = 0$ and $x = L$, the eigenfunctions and energies of the particle are, respectively, given by

$$\psi_n(x) = \frac{A_{q,n}}{\sqrt{1+\gamma_q x}} \sin\left[\frac{k_{q,n}}{\gamma_q}\ln(1+\gamma_q x)\right] \tag{41}$$

for $0 \le x \le L$, and $\psi_n(x) = 0$ otherwise, and

$$E_n = \frac{\hbar^2\pi^2\gamma_q^2 n^2}{2m\ln^2(1+\gamma_q L)}, \tag{42}$$

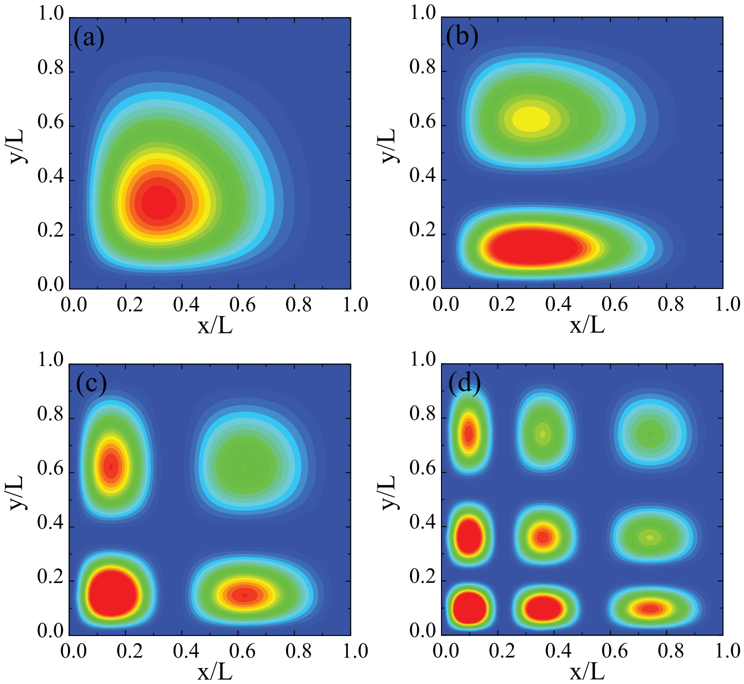

FIG. 2. Probability density contour $P(x, y) = |\psi_{n_1}(x)\psi_{n_2}(y)|^2$ for a particle confined in a bidimensional box within a generalized space with $\gamma_q L = 2$, and (a) $(n_1, n_2) = (1, 1)$, (b) $(n_1, n_2) = (1, 2)$, (c) $(n_1, n_2) = (2, 2)$, (d) $(n_1, n_2) = (3, 3)$. Color scale ranges from blue (low probabilities) to red (high probabilities).

with $A^2_{q,n} = 2\gamma_q / \ln(1 + \gamma_q L)$, $k_{q,n} = n\pi\gamma_q/\ln(1 + \gamma_q L)$ (n is a integer number). The wave function differs from those found in Refs. 7 and 15, though it is similar to that obtained in Ref. 26. Nevertheless, the energy levels are the same as those of Ref. 7.

Figure 1 shows the wave functions and their respective probability densities for the three states of lowest energy, and Figure 2 illustrates four instances of the probability density $P(x, y) = |\psi_{n_1}(x)\psi_{n_2}(y)|^2$ of a particle with position-dependent mass in a bidimensional box. It can be seen the asymmetry introduced by the position-dependent mass—the probability to find the particle around $x = 0$ increases as $\gamma_q L$ increases.

These results reduce to the usual problem of a particle confined in an infinite square well in the limit $\gamma_q \to 0$. We can see from Figure 3 that the average value of the quantum probability density approaches the classical one for large quantum numbers (here exemplified with $n = 10$), consistent with the correspondence principle.

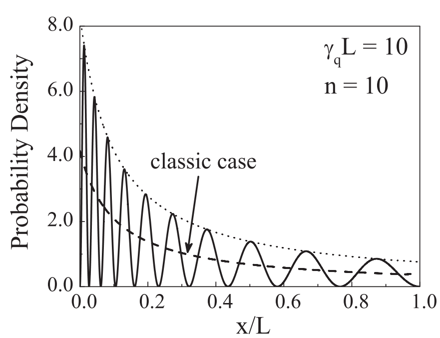

FIG. 3. Probability density of a particle confined in an infinite square well of a generalized space with $\gamma_q L = 10$ at state $n = 10$. The upper bound (dotted curve) is given by $2\gamma_q L/[(1 + \gamma_q x)\ln(1 + \gamma_q L)]$. The dashed curve is the classical case, Eq. (35).

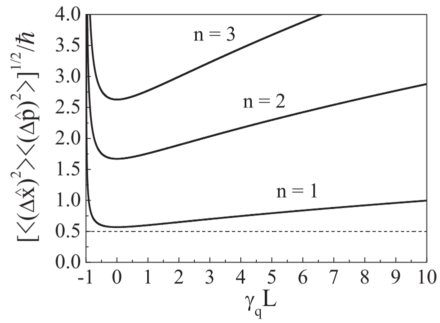

FIG. 4. $[\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle]^{1/2}/\hbar$ for different states of a confined particle, as a function of $\gamma_q L$, for states $n = 1, 2, 3$.

The expectation values of $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, and $\langle\hat{p}^2\rangle$ for the particle in a one-dimensional infinite square well are given by

$$\langle\hat{x}\rangle = \frac{\gamma_q L - \ln(1+\gamma_q L)}{\gamma_q \ln(1+\gamma_q L)} - \frac{L\ln(1+\gamma_q L)}{\ln^2(1+\gamma_q L) + (2\pi n)^2}, \tag{43a}$$

$$\begin{aligned}\langle\hat{x}^2\rangle = {} & \frac{\gamma_q^2 L^2 - 2\gamma_q L + 2\ln(1+\gamma_q L)}{2\gamma_q^2 \ln(1+\gamma_q L)} \\ & + \frac{1-(1+\gamma_q L)^2 \ln(1+\gamma_q L)}{2\gamma_q^2[\ln^2(1+\gamma_q L) + n^2\pi^2]} \\ & + \frac{2\gamma_q L \ln(1+\gamma_q L)}{\gamma_q^2[\ln^2(1+\gamma_q L) + 4n^2\pi^2]},\end{aligned} \tag{43b}$$

$$\langle\hat{p}\rangle = 0, \tag{43c}$$

$$\langle\hat{p}^2\rangle = \frac{\hbar^2 k_{q,n}^2[(1+\gamma_q L)^2 - 1]}{2(1+\gamma_q L)^2 \ln(1+\gamma_q L)}\left[1 + \frac{\gamma_q^2}{4(k_{q,n}^2 + \gamma_q^2)}\right]. \tag{43d}$$

Clearly, we can see that in the limit $n \to \infty$, Eqs. (43) coincide with Eqs. (36), obtained by the analogous problem described in the classical formalism. Moreover, one can easily show that the limit $\gamma_q \to 0$ recovers the usual results $\langle\hat{x}\rangle \to \frac{L}{2}$, $\langle\hat{x}^2\rangle \to \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$, and $\langle\hat{p}^2\rangle \to \hbar^2 k_n^2$ with $E_n = \hbar^2 k_n^2/2m$ ($k_n \equiv k_{1,n} = 2\pi n/L$).

Since the operators $\hat{x}$ and $\hat{p}$ are Hermitian and canonically conjugated, the uncertainty relation is satisfied for different values of γ_q, i.e., $\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle \geq \hbar^2/4$ (see Figure 4). Note that the product $\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle$ is minimum for $\gamma_q = 0$.

CONCLUSIONS

The modified generalized translation operator $\hat{T}_q(\varepsilon)$ (Eq. (8)) preserves the properties of that introduced by Ref. 7, Eq. (1). The corresponding generalized linear momentum operator $\hat{p}_q$, which is the generator of these translations, is Hermitian, as suggested by Ref. 15. Hermiticity permits the existence of classical analogs of the operators. The canonical transformation $(\hat{x}, \hat{p}) \to (\hat{x}_q, \hat{p}_q)$ leads the Hamiltonian of a system with position-dependent mass given by $m(x) = m/(1 + \gamma_q x)^2$ to another one of a particle with constant mass. Particularly, the classical equation of motion in the phase space may be compactly rewritten with the second dual q-derivative. We have revisited the problem of a particle confined within an infinite square well, as discussed by Refs. 7, 15, and 26. The results are consistent with the uncertainty and correspondence principles, as expected, once these dynamical variables are canonical and Hermitian.

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