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Enforcing social behavior in an Ising model with complex neighborhoods

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ABSTRACT

We consider the problem of enforcing desired behavior in a population of individuals modeled by an Ising model. Although there is a large literature dealing with social interaction models, the problem of controlling behavior in a system modeled by the Ising model seems to be an unexplored field. First, we provide and analytically characterize an optimal policy that may be used to achieve this objective. Second, we show that complex neighborhoods highly influence the decision making process. Third, we use Lagrange multipliers associated to some constraints of a related problem to identify the role of individuals in the system.

1. Introduction

A growing literature with roots in economics, sociology and statistical physics has approached individual decision-making focusing on group interaction, as opposed to individual specific determinants of behavior [1]. This literature has been used to model the process of opinion formation among individuals [2–8], selling and buying of assets [9,10], political choices [11–13], technological choices by companies [14], the preferences of consumers for a given product [15], the choice towards criminal behavior [16] and the decision of smoking [17]. Several of these models consider Ising spin chains in order to model opposite opinions. An overview of the methods of statistical and computational physics in social dynamics may be found in Refs. [18–20]. Empirical issues related to these models may be found in Ref. [21].

We consider the problem of optimally enforcing behavior of individuals or groups of individuals belonging to a population that interact with each other. This may be useful for instance to marketing companies interested in political marketing, diffusion of new products or changing habits of consumers in favor of a given company. In fact, although marketing science literature has already recognized the importance of this issue, most models in marketing science assume a priori homogeneity of the consumers [22]. Similar approaches can also be useful to interested governments that intend to fight against habits such as smoking and drug consumption or to reduce criminality in the neighborhood of a city.

Parallel to our paper, there are several attempts that are related to the problem treated here. In the field of economics, there are some works such as Refs. [23,24] that study punishment for criminal behavior of a small group of individuals that interact in very simple neighborhoods. Different from them, we work in a population explicitly modeled by a variation of the Ising model and we work with complex neighborhoods of many agents. In marketing science, there have been some attempts to characterize how consumer networks are formed such as in Ref. [25]. Therefore, our paper provides a nice

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justification for the importance of these works. Furthermore, our approach follows a growing literature mainly developed in statistical physics that intends to influence the behavior of complex networked systems. Interesting examples are the cascade control problem [26], the problem of controlling complex networks via pinning [27], the problem of target deletion of nodes in complex networks [28,29], the problem of controlling self-organized criticality [30–32] and the anti-bubble control problem [33]. Our paper is also related to a recent literature that has used optimization principles to understand the structure and dynamics of several complex systems such as complex networks [34,35] and the complex human dynamics of task execution [36,37]. In particular, in our paper, optimization principles are used in two different instances. First, optimization is used to evaluate the enforcement laws that should be used to enforce behavior in a system modeled by an Ising model. Second, optimization is used to identify the most important individuals in the system connected by complex neighborhoods. This second point is particularly linked to the literature that is concerned about the evaluation of centrality of nodes that belong to complex networks [38–40]. Although most techniques consider only topological properties of networks, using optimization techniques it is possible to define other types of centrality [41,42].

2. A model of social behavior with interactions among agents

We assume that our system is formed of N agents (or N groups of agents with the same relevant characteristics). Each of them shares one of the two opposite behaviors, denoted by $\sigma_i = \pm 1$ (for instance, smoke or not smoke), for $i = 1, \dots, N$. As we explicitly describe below, the decision of each agent towards one of these opinions depends on three different ingredients. The first ingredient is the agent's ability to support his own opinion. The second one is a given external influence towards one of the opinions (such as current media). The third one is the effect of the other agents' decisions in this agent's decision. This setup is standard in the literature of social interactions and opinion formation [1,2,4,7]. In our paper, we include the possibility that an external interested party (such as a government or a marketing company) is able to control part of the external influence that an agent is submitted to. This is the main difference from the setup of our paper to the standard one widely studied.

Let $x_i > 0$ be the so-called self-support parameter. It is the strength of the member i, i.e., a parameter that measures how difficult it is to change the opinion of this individual. The external influence on each individual of the population i is divided in two parts in our paper, namely h_i and u_i. The real number h_i represents the effect of the current campaign that includes the current efforts of interested parties (such as government or marketing companies), the current efforts of concurrent tendencies, parents education or family background resulting in a preference towards one of the opinions of the set $\{-1, 1\}$. On the other hand, the control u_i is the effect of an aggressive campaign that an interested party (such as a government or a marketing company) can provide in order to enforce a desired behavior towards one of the opinions of the set $\{-1, 1\}$ in a given individual i of the population. The idea of an aggressive campaign is that the interested party wants to increase the share of the population that is performing the desired behavior. Therefore, the external influence exerted in each agent i is given by the sum $(h_i + u_i)$, where h_i is the uncontrolled part of this influence and u_i is the controlled part. Finally, let r_{ji} be the influence of another individual j on i. Depending on the setup, the number r_{ji} may consider the social distance [43] between two individuals i and j or it can simply say that individuals i and j are connected in a given social network. In our paper, we assume that r_{ji} is 1 when the individuals are connected and 0 otherwise.

We assume here (without lack of generality) that the opinion in favor of this desired behavior is $+1$. We also assume that the opinion of agent i changes from σ_i to $\tilde{\sigma}_i$, from one period to the next one, according to

$$\tilde{\sigma}_i = \begin{cases} \sigma_i & \text{with probability } p_i^u \\ -\sigma_i & \text{with probability } 1 - p_i^u \end{cases} \tag{1}$$

where

$$p_i^u = \frac{\exp(-\beta I_i^u)}{\exp(-\beta I_i^u) + \exp(\beta I_i^u)}, \tag{2}$$

$\beta = 1/T$ is the inverse of the *social temperature*, a measure of the degree of randomness of the agents, and

$$I_i^u = -x_i - \sigma_i(h_i + u_i) - \sum_{\substack{j=1 \\ j \neq i}}^{N} r_{ji}\sigma_j\sigma_i = I_i - \sigma_i u_i \tag{3}$$

is the so-called *social impact* exerted on every individual [44,1,2,4,7]. If $I_i > 0$ then the individual is inclined to change his opinion. On the other hand, if $I_i < 0$, the opposite happens.

One may note that we have two different types of agents in this population. At a given moment, there are agents that perform the desired behavior and agents that do not. Furthermore, the aggressive campaign has different effects on different types of agents. While in the former case it must increase the chance of the agent keeping his opinion, in the latter case, the opposite should happen.

3. Optimal policy

We are looking for the optimal level of *enforcement* u_i that the interested party should introduce in the system in order to influence the behavior of an individual i of the population. Although there are several criteria that can be used to choose the vector u, we assume that the interested party will choose the aggressive campaign in order to maximize the probability of the agents in the next period performing the desired behavior. Since resources that can be used to generate the external effect u are bounded, we assume that the vector u is constrained by $\sum_{i=1}^{N} u_i = K$, where K is the total effect that the interested party can enforce with the available limited resources he has. Therefore, u_i is allowed to be positive or negative. Negative values of u_i mean that the current external effect on one individual is reduced in order to increase the effect on another individual of the population. Based on these assumptions, the problem that we are trying to solve is $\max_{u_1,\ldots,u_N} P(\tilde{\sigma}_1 = 1, \ldots, \tilde{\sigma}_N = 1|\sigma_1, \ldots, \sigma_N, u)$ subject to $\sum_{i=1}^{N} u_i = K$ where $P(\tilde{\sigma}_1 = 1, \ldots, \tilde{\sigma}_N = 1|\sigma_1, \ldots, \sigma_N, u)$ is the joint probability of $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N$ being in state 1 given $\sigma_1, \ldots, \sigma_N$. Since the distribution of $\tilde{\sigma}_i$ is independent of $\tilde{\sigma}_j$, then $P(\tilde{\sigma}_1 = 1, \ldots, \tilde{\sigma}_N = 1/\sigma_1, \ldots, \sigma_N, u) = \Pi_{i=1}^{N} P(\tilde{\sigma}_i/\sigma_1, \ldots, \sigma_N, u) \equiv \Pi_{i=1}^{N} \tilde{p}_i^u$, where

$$\tilde{p}_i^u = \exp(-\sigma_i \beta I_i^u)/2\cosh(\beta I_i^u), \tag{4}$$

for $i = 1, \ldots, n$, is by definition the probability that $\tilde{\sigma} = 1$ in the next period.

Since $\log(\cdot)$ is a strictly increasing function, this problem may be replaced by

$$\max_{u_1,\ldots,u_N} \sum_{i=1}^{N} \log \tilde{p}_i^u \quad \text{subject to} \sum_{i=1}^{N} u_i = K. \tag{5}$$

It is important to mention that the optimal policy defined by Eq. (5) is a myopic policy. It does not consider the possible future states that may occur. It only analyzes the actual state of the system and, based on this state, it makes the optimal decision in order to maximize the probability of the agents being in state 1. Having said that, even with this property, the problem presented in Eq. (5) seems to be a consistent approach. In real life, if an interested party wants to enforce some kind of policy, he would formulate a situation for a given state and keep this policy for a time, as a formulation in a two period model like here. Furthermore, this optimal policy can be analytically solved and presents several interesting properties.

Forming the Lagrangian function associated to the problem given by Eq. (5), one may show that u that solves this problem is given by the solution of the first order equations

$$\beta(1 + \sigma_i \tanh(\beta I_i^{u^\star})) = \mu^\star, \quad \text{for } i = 1, \ldots, N \tag{6}$$

$$\sum_{i=1}^{N} u_i^\star = K \tag{7}$$

where $u^\star$ is the optimal policy and $\mu^\star$ is the Lagrange multiplier associated to the equality constraint (7). Solving these equations, one may find that the optimal policy $u^\star$ and $\mu^\star$ that solve problem presented in Eq. (5) are

$$u_i^\star = \frac{I_i}{\sigma_i} + \frac{K}{N} - \overline{I/\sigma}, \quad \text{for } i = 1, \ldots, N \tag{8}$$

$$\operatorname{atanh}\left(\frac{\mu^\star - \beta}{\beta}\right) = \beta\left(\overline{I/\sigma} - \frac{K}{N}\right) \tag{9}$$

where $\overline{I/\sigma} = \frac{1}{N}\sum_{k=1}^{N} \frac{I_k}{\sigma_k}$. Besides, it is easy to show that $\frac{\partial \log(\tilde{p}_i^u)}{\mathrm{d}u_i} = -\frac{\beta^2\sigma^2}{\cosh(I^u)^2}$. Thus, one may easily prove that u_i is really a maximum of problem (5).

For the optimal policy, one gets

$$\tilde{p}_i^{u^\star} = \exp[\beta(K/N - \overline{I/\sigma})]/2\cosh[\beta(K/N - \overline{I/\sigma})]. \tag{10}$$

Therefore, the optimal policy equalizes the probabilities $\tilde{p}_i^u$, for $i = 1, \ldots, N$, for both types of agents. Moreover, if $N = 2$, then the solution of the problem is given by

$$u_1^\star = \frac{I_1}{2\sigma_1} - \frac{I_2}{2\sigma_2} + \frac{K}{2} \quad \text{and} \quad u_2^\star = \frac{I_2}{2\sigma_2} - \frac{I_1}{2\sigma_1} + \frac{K}{2}.$$

Thus, if $\sigma_1 = \sigma_2 = +1$ and $I_1 < I_2$ this implies that the probability of agent 1 keeping his position is greater than the probability of agent 2 keeping his position. Therefore, $u_1 < K/2$ in order to allow to make $u_2 > K/2$ and to increase the probability of agent 2 keeping his position. The other situations may be similarly interpreted. In fact, the intuition behind this policy is simple. If an individual is strongly inclined to the desired decision, the interested party does not need to worry about him. Furthermore, the interested party may even reduce the resources directed to him. This happens because he has a high self-support parameter in favor of the desired action, the field has a strong positive effect on him or his peers strongly affect his behavior towards the desired decision.

When $K = 0$, the role of the optimal rule (8) is only to redistribute the resources of the system. An interesting point is the effect of K in $\sum_{i=1}^{N} \log \tilde{p}_i$. Using the envelope theorem, it is easy to show that

$$\frac{\mathrm{d}\sum_{i=1}^{N} \log \tilde{p}_i^{u^\star}}{\mathrm{d}K} = \beta \left\{ 1 - \tanh\left[\beta \left(\frac{K}{N} - \overline{I/\sigma} \right) \right] \right\}. \tag{11}$$

Therefore, if we increase the amount of resources K, we also increase the joint probability of the agents being in state $\tilde{\sigma}_i = 1$, for $i = 1, \ldots, N$.

Finally, using the definition (4), we can show that

$$\log\left(\frac{\tilde{p}_i^{u^\star}}{\tilde{p}_i} \right) = \beta\sigma_i u_i^\star + \log\left(\frac{\cosh(\beta I_i)}{\cosh(\beta(I_i - \sigma_i u_i^\star))} \right) \approx \beta u_i^\star (1 + \sigma_i \tanh(\beta I_i)) \tag{12}$$

for $i = 1, \ldots, N$, where $\tilde{p}_i$ without the superscript is used to differentiate the non-enforced system (the system with $u_i = 0$, for $i = 1, \ldots, N$) from the enforced one. Therefore, if $u_i^\star > 0$, individual i in the enforced system is more likely to be in state 1.

4. Complex neighborhoods

Now we explore the effect of complex neighborhoods on the enforcement law given by Eq. (8). To play the role of complex neighborhoods, we consider the Goh et al. (GKK) scale-free networks [45], the Erdős–Renyi (ER) networks and three real networks of Swedish cities mapped into information city networks [46], namely Stockholm, Malmo and Umeå. The GKK networks and ER networks are included to take different classes of neighborhood effects into account. Although there are several networks that could play the role of a real network here, the information networks of the Swedish are included as models of real neighborhoods where the interested party may want to enforce some kind of social behavior.

The scale-free networks were built based on the algorithm provided by Ref. [45] that can be described as follows: Start with N nodes $i \in \{1, \ldots, N\}$ and assign to each of them a weight equal to $w_i = i^{-\alpha}$, where $\alpha \in [0, 1)$ is related to the degree exponent according to $\gamma = 1 + 1/\alpha$. Then select two different nodes $i, j \in \{1, \ldots, N\}$ with probability equal to the normalized weights $w_i / \sum_{k=1}^{N} w_k$ and $w_j / \sum_{k=1}^{n} w_k$, respectively, and connect them if they are not already connected. This process is repeated until mN edges are made in the system, where $2m$ is the mean degree. The Erdős–Renyi networks were built assuming that all nodes of the network have the same constant probability p of being connected. Furthermore, both GKK and ER networks were built in order to obtain the same average degree 4. The information networks of cities are networks with roads mapped to nodes and intersections to links between nodes [46].

We now proceed with our analysis based on Monte Carlo simulations using Eq. (10). More precisely, we pick up randomly an agent i and we set spin $\tilde{\sigma}_i = 1$ with probability $\tilde{p}_i$ or $\tilde{\sigma}_i = -1$ with probability $(1 - \tilde{p}_i)$, where $\tilde{p}_i$ is given by Eq. (10). The optimal policy $u^\star$ is evaluated using Eq. (8). Since, in this section, we are interested in the effect of the neighborhoods on the enforcement law given by Eq. (8), we consider here that most variables of Eq. (8) are constant. In particular, we assume that $h_i = h = 0.1$ for $i = 1, \ldots, N$. When there is an edge between nodes i and j, for $i, j = 1, \ldots, N$, $r_{ij} = r = 1$ and 0 otherwise. The parameters $x_i = 0$, for $i = 1, \ldots, N$, and $K = N$, which on the average gives an enforcement of size $K/N = 1$ for each agent.

Figs. 1 and 2 show the empirical probability distributions of the absolute values of the optimal enforcement law for the case $T = 6$ and different networks, for a typical equilibrium configuration of $\{\sigma_i\}_{i=1,\ldots,N}$. In particular, Fig. 1 shows that the GKK network with $\gamma = 2.05$ and the ER network present very different behaviors. In the inset of this figure, we show the GKK networks for $\gamma = 3.0$ and $\gamma = 2.5$. The behaviors of the absolute values of the enforcement laws that take place in these networks with larger exponents are quite similar to the one that arises in the GKK network with $\gamma = 2.05$. The only difference is the range of determination of the power law. Fig. 2 suggests that absolute values of the enforcement laws that take place in the real networks of the Swedish cities are also distributed like power laws in a very short range.

These figures suggest that the distributions of the enforcements laws for the GKK and the real networks are power laws. The same does not happen to the ER network.We have investigated these distributions in the range $T \in (0, 20]$. We have found that as long as the parameters of the agents h, r and x are kept constant with the same values, the qualitative picture presented in Figs. 1 and 2 is the same (not shown). The only clear differences are that when T increases (decreases) and the associated randomness also increases (decreases), the slopes of the power laws increase (decrease) and the intervals used to determine the exponents of the power laws are smaller (larger). Furthermore, the standard model for the study of opinion formation is the Ising model. It is well known that the Ising model presents a phase transition between an ordered state and a disordered state [47]. The same happens here at a low temperature of the interval $T \in (0, 20]$ (the specific critical temperature depends on the network topology). This fact does not affect our results. Since the role of the enforcement law is to redistribute resources, the robustness of these results is expected. In fact, in the examples presented in Figs. 1 and 2, the only source of heterogeneity that affects the agents is the position in the network and in this case the enforcement law $u^\star$

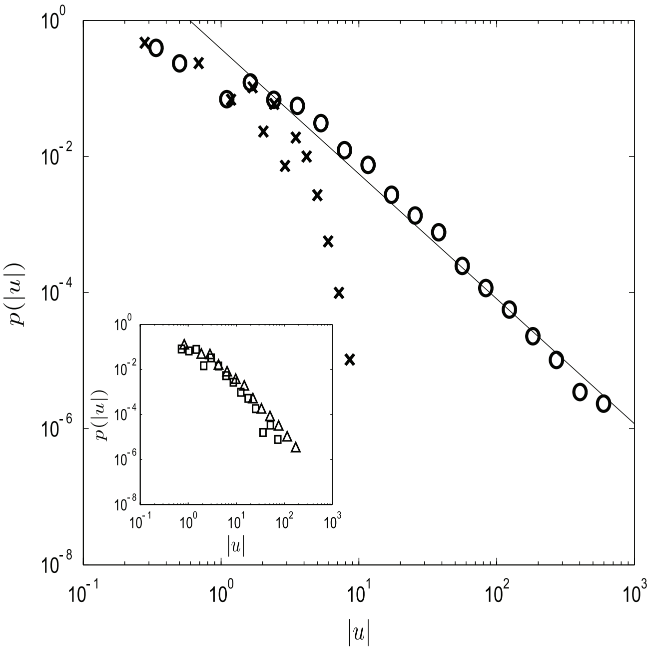

Fig. 1. Probability distribution of the absolute values of the optimal enforcement law $u^{\star}$ for computer generated networks. Points were obtained by logarithmic size bins over the whole range of $|u^{\star}|$. These simulations were performed for $T = 6$. Symbol types indicate the following values of (N, network model, η, $\mathcal{I}$), where η is the exponent of the power law and $\mathcal{I}$ is the interval used to determine this exponent: circles (10^4, GKK with $\alpha = 0.95$ and $\gamma = 2.05$, 1.84, $[10^0, 10^{3.0}]$) and crosses (10^4, ER, –, –). In the inset, squares (10^4, GKK with $\alpha = 0.5$ and $\gamma = 3.0$, 2.35, $[10^{0.1}, 10^2]$), triangles (10^4, GKK with $\alpha = 0.66$ and $\gamma = 2.5$, 2.19, $[10^0, 10^{2.5}]$).

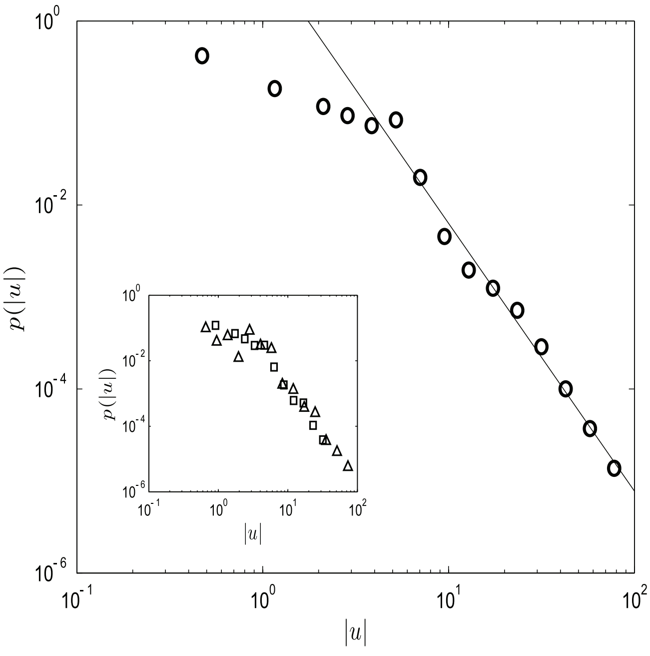

Fig. 2. Probability distribution of the absolute values of the optimal enforcement law $u^{\star}$ for information networks of cities. Points were obtained by logarithmic size bins over the whole range of $|u^{\star}|$. These simulations were performed for $T = 6$. Symbol types indicate the following values of (N, network model, η, $\mathcal{I}$), where η is the exponent of the power law and $\mathcal{I}$ is the interval used to determine this exponent: circles (10 713, Stockholm city, 2.91, $[10^{0.5}, 10^{2.0}]$). In the inset, squares (3735, Malmo, 2.22, $[10^{-0.01}, 10^{2.0}]$), triangles (566, Umeå, 2.68, $[10^0, 10^{1.6}]$).

is strongly dependent on the topology of the neighborhood that the agents interact to each other. Therefore, representative agent models, models where there is only one agent representing on average the entire population (a kind of mean field approach), commonly used in economics and social science do not work here. This scenario has also arisen in other complex system distributed control problems such as in the cascade control problem [26], the problem of target deletion of nodes in complex networks [28,29], the problem of controlling self-organized criticality [30–32] and the anti-bubble control [33]. In all these above-cited control problems, it is essential to consider the spatial nature of the problem when choosing the control strategy.

5. How may one identify the role of each individual in the network?

Another interesting point is how to identify the role of each individual in the network. We have seen that the role of the optimal enforcement law is to equalize the probability that the agents in the system behave as desired. In order to equalize the system, the optimal enforcement law transfers external resources or resources from the agents that are already likely to perform the desired behavior to the ones that need some enforcement. In this section we present a method to identify the role of each agent in the system, i.e., we want to identify the most important providers and beneficiaries of resources in the system. This problem is very related to the problem of determining centrality of nodes in complex networks [38–40]. However, as we will see below, the problem here depends on other characteristics of the agents not uniquely determined by their relations in the network. Since we have the vector of optimal enforcement laws given by Eq. (8), one way to test the quality of the method is to evaluate the correlation between the solution provided by the method and the vector of optimal enforcement laws. If there is a high correlation between these two vectors, this implies that the method works. Otherwise, the method does not provide a good answer to this question.

Since we have considered that the neighborhoods of the agents are provided by complex networks, one question that arises is if the vector of the degree of the nodes $\kappa = (\kappa_1, \ldots, \kappa_N)$ is a good proxy to identify the roles of these individuals in the system. A simple way to answer this question is to evaluate the correlation coefficient ρ between the vector of optimal enforcement laws $u^\star$ and the vector of degrees of the nodes κ. In order to test this issue, we assume a setup with $T = 1, h_i = h = 0$ for $i = 1, \ldots, N$ and the other parameters are the same ones used in Section 4. Furthermore, if we consider that the neighborhood of the agents was built using a GKK network with $\gamma = 2.05$, we find that the Spearman correlation between the vector of optimal enforcement laws and the vector of degrees is $\rho = -0.99$. The Spearman correlation coefficient was used since it assesses how well the relationship between two variables can be described using a monotonic function.[1] This result means that more connected nodes are associated to smaller enforcement laws. This can be easily explained. If there is a global field in the direction of the desired behavior, most individuals are expected to show the desired behavior and consequently the most connected individuals will receive a larger reinforcement in this direction from their peers. Therefore, in this context, these individuals have a singular role in the system, since they are the providers of resources. The interested party reduces the optimal enforcement law directed to these individuals in order to increase the probability of the less connected individuals choosing the desired option. Therefore, in this situation we can assert that the degree of the individual is an easy way to identify the role of each individual in the network. Consider now that we keep the same parameters of the simulation presented above, but we increase the temperature for $T = 4$. As expected the Spearman correlation coefficient between the values of the enforcement laws and degrees decreases (in terms of absolute value) to $\rho = -0.84$, since the randomness of the system increases. In the above-mentioned situation we have considered that all individuals in the system have the same parameters, but different neighborhoods. Therefore, the results that we are finding is that, excluding the randomness of the system, in this case all that matters is how well connected the individual is. What happens if the self-support parameter x_i of each individual i is independently distributed? Is the degree of each individual still a good way to characterize each individual in the network? As we show below, we will provide a negative answer to this question.

Here, we suggest that we could identify these individuals using some kind of information inherited from the optimization problem presented by Eq. (5) that takes place in the complex network. To do this, we define an associate optimization problem where the only difference from the problem presented in Eq. (5) is that this new problem has a set of additional positivity constraints, namely $0 \leq u_i$ for $i = 1, \ldots, N$. The idea here is to use the Lagrange multipliers associated to these positivity constraints $u_i \geq 0$, for $i = 1, \ldots, N$, in order to identify the role of each individual in the system. In fact, Lagrange multipliers associated to positivity constraints measure how much the optimum of the system would increase, if the constraint were relaxed [48]. In our context, this means that the set of Lagrange multipliers $\{\lambda_i\}_{i=1,\ldots,N}$ associated to the constraint $0 \leq u_i$ measures the effect of relaxing the constraint $0 \leq u_i$ to $\delta_i \leq u_i$, for $i = 1, \ldots, N$, where δ_i is infinitesimally less than zero. Therefore, the individuals who have larger Lagrange multipliers are the most important resource providers (see the example below in order to get the big picture). It is worth mentioning that this idea reminds closely the one considered in Ref. [49] that uses Lagrange multipliers to set personalized intraday monetary policies in order to reduce idleness of money in a payment system defined in a Banach space.

Unfortunately, this new problem given by the problem defined in Eq. (5) with the additional positivity constraints cannot be solved analytically. Even numerically it can be hardly solved for the case of low heterogeneity and a very small number of individuals. One way to circumvent this difficulty is to approximate this problem by a linear programming problem. In order to do this, we use the linear approximation of $\log \tilde{p}_i$ (defined in Eq. (4)) given by $\log \tilde{p}_i \approx -\beta\sigma_i I_i - \log(2\cosh(\beta I_i)) +$

[1] We use the Spearman correlation since we are not concerned about the values of these variables, but about their ranks. The Spearman correlation coefficient is actually the Pearson correlation coefficient (the most standard measure of correlation) between the ranked variables.

$\beta u_i + \beta \sigma_i u_i \tanh(\beta I_i)$. Therefore, using this linear approximation without the constant terms (since they do not influence the optimization process), we define the problem

$$\max_{u_1,\ldots,u_n} \sum_{i=1}^{n} \beta u_i [1 + \sigma_i \tanh(\beta I_i)] \tag{13}$$

subject to

$$\sum_{i=1}^{N} u_i = K \tag{14}$$

$$0 \le u_i \quad \text{for } i = 1, \ldots, N \tag{15}$$

where the solution of this problem is a vector $u^o = (u_1^o, \ldots, u_n^o)$, a Lagrange multiplier μ^0 associated to the equality constraint and a vector of Lagrange multipliers $\lambda^o = (\lambda_1^o, \ldots, \lambda_N^o)$ associated to the positivity constraints. We have used the superscript "o" instead of "$\star$" (previously used to define the solutions of the problem (5)) in order to stress that they are solutions of different problems.

First, note that this problem is well specified, since the objective function is bounded due to the boundedness of u_i for $i = 1, \ldots, N$. Second, although the objective function of this problem does not represent probabilities any more, it still has an interesting taste. Note, for instance, that if $\sigma_i > 0$ and $I_i \ll 0$, then less enforcement is applied to the individual i. Third, we do not intend to use the solution of this problem to enforce the behavior of agents, but only to identify the role of each individual in the system. However, if one claims that in some situations it makes no sense to use negative enforcement laws, the solution u^o can be a useful approximation. Fourth, in order to evaluate u^0, μ^0 and λ^0, as in the case of problem (5), we only need the parameters of the system and the configuration of the agents in a given time. We do not need any information about $u^\star$! Fifth, the quality of the solution depends on the quality of the linear approximation that is valid for small values of enforcement laws.

In order to get the big picture of the idea, consider that we want to solve this problem for a system with $N = 3$ individuals, $K = 1$, $\beta[1+\sigma_1 \tanh(\beta I_1)] = 1$, $\beta[1+\sigma_2 \tanh(\beta I_2)] = 2$ and $\beta[1+\sigma_3 \tanh(\beta I_3)] = 3$. Therefore, this problem with these specifications is given by

$$\max_{u_1,u_2,u_3} u_1 + 2u_2 + 3u_3$$

subject to

$$u_1 + u_2 + u_3 = 1, \quad u_1 \ge 0,\ u_2 \ge 0,\ u_3 \ge 0.$$

It is easy to show that the solution of this problem is given by $(u_1, u_2, u_3, \lambda_1, \lambda_2, \lambda_3, \mu) = (0, 0, 1, 2, 1, 0, 3)$. Note that if we could have $u_1 = \delta_1 < 0$ and $u_2 = \delta_2 < 0$, we would be able to increase the maximum of the function $u_1 + 2u_2 + 3u_3$ by $-2\delta_1 - \delta_2$ units. Furthermore, the effect of reducing the value of u_1 by $-\delta_1$ units is larger than the effect of reducing the value of u_2 by $-\delta_2$ units. Therefore, individual 1 is a better resource provider than individual 2. Thus, $\lambda_1 > \lambda_2$. This very simple example shows that it is possible to identify the best resource providers of the network using the information of the Lagrange multipliers associated to the positivity constraints.

Let us now test these ideas in the GKK network. We use the same setup of the simulation presented in the beginning of this section with $T = 1$, but here $K = 1000$ (which on the average gives an enforcement of size $K/N = 0.1$ for each individual) and the self-support parameter $x_i = c\xi$, where $\xi \in [0, 1]$ is a uniform random variable and c is a constant. We use a smaller value of K in order to increase the heterogeneity among the individuals and to make a more precise valuation of the method used to identify the role of each node in the system. We have used $c \in [0, 50]$, because if we use c larger than that, then the effect of the enforcement is totally insignificant. In fact, note that when $c = 50$, 50% of the individuals have on the average the self-support parameter larger than 25, which on the average is 250 times the size of the average enforcement $K/N = 0.1$. Fig. 3 shows the effect of c on the Spearman correlation coefficients between the vector of the degrees of the nodes κ and the vector of Lagrange multipliers of the positivity constraints λ^o, the Spearman correlation coefficients between the vector of the degrees of the nodes κ^o and the vector of optimal enforcement laws $u^\star$ and the correlation coefficients between the vector of Lagrange multipliers of the positivity constraints λ^o and the vector of optimal enforcement laws $u^\star$. From this figure, we are able to answer the question posed in the beginning of this section. When c increases, the degree is no longer a good proxy for identifying the role of each individual in the system. However, we are able to determine the best resource providers (and also beneficiaries) using the vector of Lagrange multipliers λ^o, with perfect correlation for a large range. One may also see that if $c > 20$, the correlation between the vector of Lagrange multipliers and the best resource providers is not perfect anymore. This is not due to a fault of the method, but, as mentioned above, the average enforcement law applied to each individual is almost insignificant. In conclusion, the vector of Lagrange multipliers λ^o may be used in the same way we are used to applying the centrality measures very common in complex networks—just to identify the kinds of individuals in the system, without necessarily thinking in enforcing the behavior of the individuals.

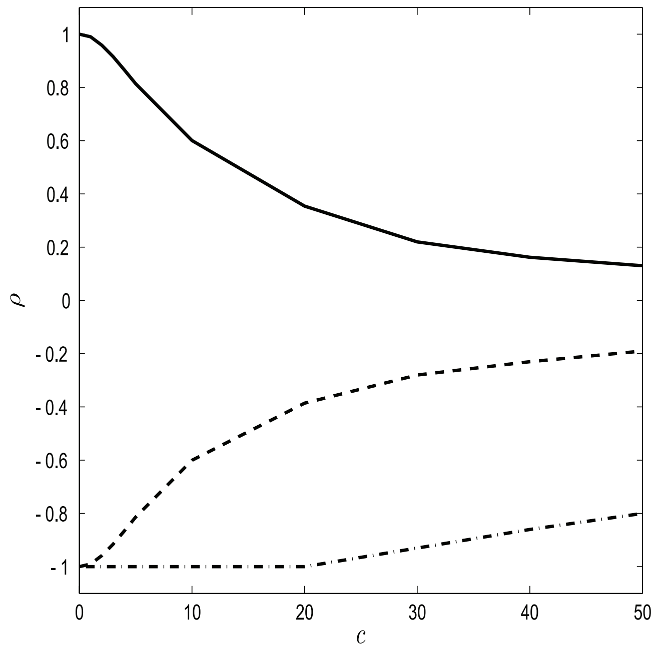

Fig. 3. This figure shows the Spearman correlation coefficients ρ between two vectors as a function of c. Solid curve: The Spearman correlation coefficients between the vector of the degrees of the nodes κ and the vector of Lagrange multipliers of the positivity constraints λ. Dashed curve: The Spearman correlation coefficients between the vector of the degrees of the nodes κ and the vector of optimal enforcement laws $u^{\star}$. Dotted–dashed curve: The correlation coefficients between the vector of Lagrange multipliers of the positivity constraints λ and the vector of optimal enforcement laws $u^{\star}$.

6. Summary and further research

We have considered the problem of enforcing social behavior in a system modeled by the Ising model. We have provided a method to control the Ising model suggesting that this can be controlled by simple resource distribution rules. We have also studied the effects of complex neighborhoods in enforcement laws providing a fair justification for studying problems of social interactions using complex networks. Finally, we have used Lagrange multipliers to identify the role of each individual in the system.

Further research may consider concurrent techniques to solve similar control problems taking place in Ising models. In particular, we have assumed that we have information about all agents in the network and we are able to control the behavior of all of them. There are two interesting cases that have not been considered here. The first one is to consider that although we have complete information about all agents in the network, we control only a small fraction of them and expect that the manipulation of these agents are enough to influence the behavior of the entire network. This problem is a particular case of the problem studied in this paper and it is actually very related to the problem of target deletion of nodes in complex networks [28,29]. The second one is to assume that we cannot control all agents of the network and we do not have information about all agents. This problem is much more complicated and one should consider a kind of robust strategy in order to deal with the uncertainty that arises in this setup. In our paper, we have considered that our objective is to maximize the probability of the agents will perform the desired behavior. There are other possibilities for the performance index such as to maximize the magnetization of the system that can be considered in the future. Another important point is that we have presented here an enforcement strategy that was derived considering only the efficiency of the distribution of resources. There may be some particular situations where ethical or incentive issues may require the inclusion of other criteria. Moreover, if one assumes that performing a desirable behavior is a public good, some agents could be willing to pay for that kind of behavior. The contribution of each agent could follow a demand revealing mechanism such as the one provided by the Groves–Clarke method [50]. Therefore, the government could supply a Pareto optimal quantity of public goods based on this information and constraint K could be make endogenous.

In this paper, we have considered only few examples of networks as models of the neighborhoods of agents. A future work may also study the effect of different classes of networks in the problem. One may investigate explicitly the effect of hierarchical or assortative (disassortative) neighborhoods on enforcement laws.

Another interesting path is to adapt the setup presented here in order to deal with the particularities of a specific area such as control of criminality or marketing science.

Finally, the idea of identifying the role of a node in the system using Lagrange multipliers can be useful in other networks that present other kinds of fluxes such as biological networks [29]. In some situations, the problem may be naturally linear and linear approximation will not be necessary.

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