AIP | Journal of Mathematical Physics

Symmetric deformed binomial distributions: An analytical example where the Boltzmann-Gibbs entropy is not extensive

H. Bergeron, E. M. F. Curado, J. P. Gazeau, and Ligia M. C. S. Rodrigues

Citation: Journal of Mathematical Physics **57**, 023301 (2016); doi: 10.1063/1.4939917
View online: http://dx.doi.org/10.1063/1.4939917
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/57/2?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **57**, 023301 (2016)

Symmetric deformed binomial distributions: An analytical example where the Boltzmann-Gibbs entropy is not extensive

H. Bergeron,[1,a)] E. M. F. Curado,[2,3,b)] J. P. Gazeau,[2,4,c)] and Ligia M. C. S. Rodrigues[2,d)]

[1]*Univ Paris-Sud, ISMO, UMR 8214, 91405 Orsay, France*
[2]*Centro Brasileiro de Pesquisas Fisicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brazil*
[3]*Instituto Nacional de Ciência e Tecnologia–Sistemas Complexos, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brazil*
[4]*APC, UMR 7164, Univ Paris Diderot, Sorbonne Paris Cité, 75205 Paris, France*

(Received 6 June 2015; accepted 30 December 2015; published online 20 January 2016)

Asymptotic behavior (with respect to the number of trials) of symmetric generalizations of binomial distributions and their related entropies is studied through three examples. The first one has the q-exponential as the generating function, the second one involves the modified Abel polynomials, and the third one has Hermite polynomials. We prove analytically that the Rényi entropy is extensive for these three cases, i.e., it is proportional (asymptotically) to the number n of events and that q-exponential and Hermite cases have also extensive Boltzmann-Gibbs. The Abel case is exceptional in the sense that its Boltzmann-Gibbs entropy is not extensive and behaves asymptotically as the square root of n. This result is obtained numerically and also confirmed analytically, under reasonable assumptions, by using a regularization of the beta function and its derivative. Probabilistic urn and genetic models are presented for illustrating this remarkable case. © *2016 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4939917]

I. INTRODUCTION

The content of our previous papers[1–4] was devoted to a comprehensive study of discrete distributions generalizing the familiar binomial distributions. The generalization consists in substituting the ordinary integers on which is based the binomial distribution by arbitrary sequences of positive numbers. These distributions can be symmetrical or asymmetrical. Such generalizations introduce correlations among the events, as was shown through an example in Section 3.1.1 in Ref. 2. The study concerned the positiveness of those formal distributions in order to view them as having a real probabilistic content. We have given many examples, which run from Delone sequences, q-sequences, sequences based on family of polynomials (modified Abel, Hermite, etc.). A key point of our works was to display manageable generating functions. The existence of such functions allows to easily control positiveness and makes a series of computations easier. Hence, we have shown in the above references a palette of interesting properties. Nevertheless, except in one case, we did not explore systematically their asymptotic behaviors, their associated entropies (Shannon or Boltzmann-Gibbs (BG), Tsallis, Rényi, etc.), and related questions like extensiveness.

In the present article, we analyze the asymptotic behaviors of three different generalized distributions[4] and their respective Boltzmann-Gibbs and Rényi entropies. In particular, we focus our study on the extensiveness properties of the latter and we present a special case where the

a)E-mail: herve.bergeron@u-psud.fr
b)E-mail: evaldo@cbpf.br
c)E-mail: gazeau@apc.univ-paris7.fr
d)E-mail: ligia@cbpf.br

extensive entropy is the Rényi and not the Boltzmann-Gibbs one. We recall that a statistical entropy is extensive (respectively, asymptotically extensive) if it is proportional (respectively, asymptotically proportional) to the number n of events (respectively, at large n). The generalized probability distributions mentioned above are denoted in this paper by

$$\mathcal{P} = \left(\mathfrak{p}_1^{(n)}, \mathfrak{p}_2^{(n)}, \ldots \mathfrak{p}_n^{(n)}\right). \tag{1.1}$$

In the binomial distribution, these probabilities can be written as $\mathfrak{p}_k^{(n)}(\eta) = \binom{n}{k}\eta^k(1-\eta)^{n-k}$, where η is a parameter belonging to the interval $[0,1]$. Due to symmetry, the multiplicity of states in the generalized case is the same as for the binomial distribution. When we deform the binomial in the way of Refs. 1 and 2, correlations among events are introduced. In our evaluations of entropies, we adopt a "microscopic" point of view by ignoring the multiplicity, i.e., we look at the probability of a particular event with k "wins" and $n-k$ "losses," $\varpi_k^{(n)} \equiv \mathfrak{p}_k^{(n)}/\binom{n}{k}$.

The first statistical entropy studied here is the BG or Shannon[5,6] entropy,

$$S_{\mathrm{BG}} = -\sum_{k=0}^{n}\binom{n}{k}\varpi_k^{(n)}\log\varpi_k^{(n)}. \tag{1.2}$$

The second one is the Rényi entropy $S_{\mathrm{Re};q}$,[7] which is a deformation of (1.2), $S_{\mathrm{Re};q} \to S_{\mathrm{BG}}$ as $q \to 1$, at least for finite n,

$$S_{\mathrm{Re};q} = \frac{1}{1-q}\log\left[\sum_{k=0}^{n}\binom{n}{k}\left(\varpi_k^{(n)}\right)^q\right]. \tag{1.3}$$

In this article, we prove that the Rényi entropy is extensive for these three cases and that the former (q-exponential) and the latter (Hermite) have also extensive Boltzmann-Gibbs. On the other hand, the second case (modified Abel) is exceptional since its Boltzmann-Gibbs entropy is not extensive and behaves asymptotically as the square root of the number of events. This result is obtained numerically and also confirmed analytically, under reasonable assumptions, by regularizing the beta function and its derivative.

The organization of the paper is the following. In Section II, the necessary background for symmetric deformations of the binomial distribution and issued from Ref. 4 is recalled. In Section III, we complete this material with a study of asymptotic behaviors of expectation values calculated with these distributions. In Section IV, we present three examples of generalized symmetric distributions, having, respectively, as generating functions, the so-called q-exponential, in Subsection IV A, the Lambert function, the most interesting case, in Subsection IV B and, finally, the function $\exp\left(t + \frac{a}{2}t^2\right)$ in Subsection IV C. The Lambert function gives rise to a generalized distribution which depends on a modified version of Abel polynomials. Due to the importance of this case within the question of extensivity of entropies, we present in Subsection IV B 3 a an urn-like probabilistic model based on counting of words made with letters picked in several alphabets. This model is quite elaborate and we illustrate it in Subsection IV B 3 a a with a more realistic example extracted from Genetics. In Subsection IV C, our example involves Hermite polynomials. In Section V, we analyze the behavior of the Boltzmann-Gibbs and Rényi entropies concerning their extensivity properties for the three generalized distributions presented here. In Subsection V A, we derive those two entropies for the generalized distributions from the q-exponential and find that not only Boltzmann-Gibbs is extensive but also the Rényi one. Subsection V B is devoted to our main result: for the generalized distribution given by modified Abel polynomials, the entropy which is asymptotically extensive is not Boltzmann-Gibbs, but instead the Rényi one, and its asymptotic behavior does not depend on the Rényi parameter. Finally, Subsection V C is devoted to the analysis of Boltzmann-Gibbs and Rényi entropies for our third generalized distribution, involving Hermite polynomials. With this case, we return to the standard situation for which both Boltzmann-Gibbs and Rényi are extensive. In Section VI, we present our conclusions and final comments. The Appendix is devoted to regularization methods employed in this work in order to get rid of singularities as the number n of events goes to infinity. These regularizations are requested in the study of the Abel case and concern the function beta and its derivatives when their integral representations become divergent.

II. SYMMETRIC DEFORMATIONS OF BINOMIAL DISTRIBUTIONS

We remind in this section the notations and main results of Ref. 4.

Let $\mathcal{X} = (x_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers x_n for $n > 0$ and $x_0 = 0$. The "factorial" of x_n is defined as $x_n! = x_1\, x_2 \ldots x_n$, $x_0! \stackrel{\text{def}}{=} 1$, and from it, we build the binomial coefficient

$$\binom{x_n}{x_k} := \frac{x_n!}{x_{n-k}!x_k!}\,.$$

We now associate to $\mathcal{X}$ the formal distribution

$$\mathfrak{p}_k^{(n)}(\eta) = \binom{x_n}{x_k}\, q_k(\eta)q_{n-k}(1-\eta), \tag{2.1}$$

where the $q_k(\eta)$ are polynomials of degree k and the $\mathfrak{p}_k^{(n)}(\eta)$ are constrained by the *normalization* condition

$$\forall n \in \mathbb{N}, \quad \forall \eta \in [0,1], \quad \sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(\eta) = 1, \tag{2.2}$$

and by the *non-negativeness* condition

$$\forall n,k \in \mathbb{N}, \quad \forall \eta \in [0,1], \quad \mathfrak{p}_k^{(n)}(\eta) \geq 0. \tag{2.3}$$

The normalization implies

$$\forall \eta \in [0,1], \quad \mathfrak{p}_0^{(0)}(\eta) = q_0(\eta)q_0(1-\eta) = 1 \Rightarrow q_0(\eta) = \pm 1.$$

From now on, we keep the choice $q_0(\eta) = 1$. This implies

$$\forall n \in \mathbb{N}, \quad \forall \eta \in [0,1], \quad \mathfrak{p}_0^{(n)}(\eta) = q_n(1-\eta), \mathfrak{p}_n^{(n)}(\eta) = q_n(\eta).$$

Therefore, the non-negativeness condition is equivalent to the non-negativeness of the polynomials q_n on the interval $[0,1]$. The quantity $\mathfrak{p}_k^{(n)}(\eta)$ can be interpreted as the probability of having k wins and $n-k$ losses in a sequence of *correlated* n trials. Besides, as we recover the invariance under $k \to n-k$ and $\eta \to 1-\eta$ of the binomial distribution, no bias in the case $\eta = 1/2$ can exist favoring either win or loss.

We now associate to the sequence $\mathcal{X}$ a deformed "exponential" defined as the power series

$$\mathcal{N}(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!} \equiv \sum_{n=0}^{\infty} a_n t^n, \quad x_n = a_{n-1}/a_n, \tag{2.4}$$

which is supposed to have a non-vanishing radius of convergence. Hence, $\mathcal{N}$(t) is an element of Σ defined as the set of power series $\sum_{n=0}^{\infty} a_n t^n$ possessing a non-vanishing radius of convergence and verifying $a_0 = 1$ and $\forall n \geq 1$, $a_n > 0$.

Starting from $\mathcal{N}(t) \in \Sigma$ and $\eta \in [0,1]$, we consider the series $\mathcal{N}(t)^\eta$. It is easy to prove from $\mathcal{N}(t) = \mathcal{N}(t)^\eta\, \mathcal{N}(t)^{1-\eta}$ that it is a generating function for polynomials q_n obeying (2.1) and (2.2),

$$\forall \eta \in [0,1], \quad G_{\mathcal{N},\eta}(t) := \mathcal{N}(t)^\eta = \sum_{n=0}^{\infty} \frac{q_n(\eta)}{x_n!} t^n. \tag{2.5}$$

More precisely, the polynomials q_n issued from (2.5) have the following properties:

(a) $q_0(\eta) = 1$, $q_1(\eta) = \eta$ and more generally,

$$\forall n \in \mathbb{N}\,, \forall \eta \in [0,1], \; q_{n+1}(\eta) = \eta\, \frac{x_{n+1}}{n+1} \times \sum_{k=0}^{n} \binom{x_n}{x_k} \frac{n-k+1}{x_{n-k+1}}\, q_k(\eta-1). \tag{2.6}$$

(b) The q_n's are polynomials of degree n obeying

$$\forall n \in \mathbb{N}, \; q_n(1) = 1, \text{ and } \forall n \neq 0, \; q_n(0) = 0,$$

and they fulfill the normalization condition.

(c) The q_n's fulfill the functional relation

$$\forall z_1, z_2 \in \mathbb{C}, \forall n \in \mathbb{N}, \sum_{k=0}^{n} \binom{x_n}{x_k} q_k(z_1) q_{n-k}(z_2) = q_n(z_1 + z_2). \tag{2.7}$$

These polynomials, suitably normalized, are of binomial type.[8] Indeed, with the definition $\tilde{q}_n(\eta) = \frac{n!}{x_n!} q_n(\eta)$, we have the property

$$\tilde{q}_n(z_1 + z_2) = \sum_{k=0}^{n} \binom{n}{k} \tilde{q}_k(z_1) \tilde{q}_{n-k}(z_2)$$

which defines such polynomials.

Since we already know that $q_0(\eta) = 1$ and $\forall\, n \neq 0, q_n(0) = 0$, the non-negativeness condition is equivalent to specify that for any $\eta \in]0,1]$, $q_n(\eta) > 0$ and then the function $t \mapsto \mathcal{N}(t)^\eta$ belongs to Σ. Defining Σ_0 as the set of entire series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ possessing a non-vanishing radius of convergence and verifying the conditions $a_0 = 0$, $a_1 > 0$ and $\forall n \geq 2$, $a_n \geq 0$, it was proved in Ref. 4 that

$$\Sigma_+ := \{\mathcal{N} \in \Sigma \,|\, \forall \eta \in [0,1), \forall n \geq 0,\, q_n(\eta) > 0\,\} = \{e^F \,|\, F \in \Sigma_0\} \tag{2.8}$$

is the set of deformed exponentials such that the generating functions $G_{\mathcal{N},\eta}(t)$ solve the non-negativeness problem.

III. ASYMPTOTIC ESTIMATES

Since we are concerned in this paper with asymptotic behaviors of distributions and their entropies, it is opportune to give already some hints about the asymptotic behavior at large n of sums (expectation values) of the type

$$\sum_{k=0}^{n} \phi_n(k) \mathfrak{p}_k^{(n)}(\eta), \tag{3.1}$$

where $\mathbb{R} \ni x \to \phi_n(x)$ are C^∞ functions. These preliminary results will be used for implementing our regularization procedures. To obtain the dominant term, we first determine the asymptotic expression $\rho_n(x)$ of $\mathfrak{p}_{nx}^{(n)}(\eta)$ at large n (with $x \in (0,1)$), and then replace the sum $\sum_{k=0}^{n}$ with the integral $n\int_0^1 \mathrm{d}x$. We obtain

$$\sum_{k=0}^{n} \phi_n(k) \mathfrak{p}_k^{(n)}(\eta) \underset{\text{large } n}{\sim} n \int_0^1 \phi_n(nx) \rho_n(x,\eta) \mathrm{d}x. \tag{3.2}$$

In the case of modified Abel polynomials, which is examined in Subsections IV B and V B, the above integral $\int_0^1$ is improper: it diverges at one or at both endpoint(s). This happens because, at the endpoint(s), we are violating the condition(s) $nx \gg 1$ or/and $n(1-x) \gg 1$. A regularization procedure is needed and is detailed in the Appendix. It involves two regularization parameters $\epsilon_1(n)$ and $\epsilon_2(n)$, which vanish as n goes to infinity. Estimate (3.2) becomes

$$\sum_{k=0}^{n} \phi_n(k) \mathfrak{p}_k^{(n)}(\eta) \underset{\text{large } n}{\sim} n \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \phi_n(nx) \rho_n(x,\eta) \mathrm{d}x. \tag{3.3}$$

In the Appendix, we give the expressions of $\epsilon_1(n)$ and $\epsilon_2(n)$ by using our previous result Eq. (32) in Ref. 4 concerning the expectation value of the k variable,

$$\sum_{k=0}^{n} k\, \mathfrak{p}_k^{(n)}(\eta) = n\eta. \tag{3.4}$$

Then, the general regularized expression in Eq. (3.3) is partially validated (in the Abel case) by comparing the asymptotic behavior of elementary expressions obtained in Ref. 4 (like the expectation values $\langle k^2 \rangle_n$ and $\langle k(n-k) \rangle_n$) with their estimates given in Eq. (3.3). However, we are unable to prove analytically the validity of our regularization procedure for functions ϕ_n which have singular behavior at the endpoints 0 and 1 of the integral. Indeed, in the Abel case considered in Subsection V B, we have to deal with functions which have a mild (logarithmic) singularity in the endpoints. In this particular case, we rely upon the numerical validation.

IV. THREE CASES OF GENERALIZED SYMMETRIC DISTRIBUTIONS

A. Symmetric distribution from "q-exponential"

1. Definition and probabilistic model

We consider here the following family of functions belonging to Σ_+:

$$\mathcal{N}(t) = \left(1 - \frac{t}{\alpha}\right)^{-\alpha}, \quad \alpha > 0 \tag{4.1}$$

that are q-exponentials in the sense that $e_{\mathsf{q}}(x) = [1 + (1 - \mathsf{q})x]^{(1/(1-\mathsf{q}))}$, where the parameter $\mathsf{q} = 1 + 1/\alpha$ with the notations in Ref. 9. We first note that if $\alpha \to \infty$, then $\mathcal{N}(t) \to e^t$, i.e., we return to the ordinary binomial case. The corresponding sequence is bounded by α and given by

$$x_n = \frac{n\alpha}{n + \alpha - 1}, \quad \lim_{n\to\infty} x_n = \alpha. \tag{4.2}$$

For the factorial, we have

$$x_n! = \alpha^n \frac{\Gamma(\alpha) n!}{\Gamma(n + \alpha)} = \frac{\alpha^n n!}{(\alpha)_n} = \frac{\alpha^n}{\binom{n+\alpha-1}{n}}, \tag{4.3}$$

where $(z)_n = \Gamma(z + n)/\Gamma(z)$ is the Pochhammer symbol. The corresponding polynomials are given by

$$q_n(\eta) = \frac{\Gamma(\alpha)}{\Gamma(n + \alpha)} \frac{\Gamma(n + \alpha\eta)}{\Gamma(\alpha\eta)} = \frac{(\alpha\eta)_n}{(\alpha)_n} \tag{4.4}$$

and satisfy the recurrence relation

$$q_n(\eta) = \frac{n + \alpha\eta - 1}{n + \alpha - 1}\, q_{n-1}(\eta), \quad \text{with } q_0(\eta) = 1. \tag{4.5}$$

The distribution $\mathfrak{p}_k^{(n)}(\eta)$ defined by these polynomials is given by

$$\mathfrak{p}_k^{(n)}(\eta) = \binom{n}{k} \frac{\Gamma(\alpha)}{\Gamma(\eta\alpha)\Gamma((1-\eta)\alpha)} \frac{\Gamma(\eta\alpha + k)\Gamma((1-\eta)\alpha + n - k)}{\Gamma(\alpha + n)} \tag{4.6}$$

$$= \frac{\binom{\eta\alpha + k - 1}{k}\binom{(1-\eta)\alpha + n - k - 1}{n - k}}{\binom{\alpha + n - 1}{n}}. \tag{4.7}$$

This is precisely the Pólya distribution,[10] also called "Markov-Pólya" or "inverse hypergeometric" and more. It was considered by Pólya (1923) in the following urn scheme.[11] From a set of b black balls and r red balls contained in an urn, one extracts one ball and returns it to the urn together with c balls of the same color. The probability to have in the urn k black balls after the nth trial is given by ratio (4.7) with

$$\eta = \frac{b}{b + r}, \qquad \alpha = \frac{b + r}{c}, \tag{4.8}$$

which holds for rational parameters η and α. In this notation, distribution (4.6) reads, in terms of Pochhammer symbol,

$$\mathfrak{p}_k^{(n)}(b,c,r) = \binom{n}{k} \frac{\left(\frac{b}{c}\right)_k \left(\frac{r}{c}\right)_{n-k}}{\left(\frac{b+r}{c}\right)_n}. \tag{4.9}$$

We notice that if we take the medium value $\eta = 1/2$ and redefine the parameters according to $\alpha \to 2\nu$, $n \to N$, and $k \to n$ in the distribution given by Eq. (4.6), we recover the distribution r_n^N studied in Ref. 12, see Eqs. (4) and (10) therein, within the framework of the Laplace-de Finetti representation.

2. Asymptotic behavior at large n

Let us now study the asymptotic behavior of (4.6) at large n. The probability distribution is given by

$$\begin{aligned}\mathfrak{p}_k^{(n)}(\eta) &= \binom{n}{k} \frac{\Gamma(\alpha)}{\Gamma(\eta\alpha)\Gamma((1-\eta)\alpha)} \frac{\Gamma(\eta\alpha+k)\Gamma((1-\eta)\alpha+n-k)}{\Gamma(\alpha+n)} \\ &= \binom{n}{k} \frac{B(\eta\alpha+k,(1-\eta)\alpha+n-k)}{B(\eta\alpha,(1-\eta)\alpha)},\end{aligned}$$

where $0 \le \eta \le 1$, $\alpha > 0$, and $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(q+p)$ is the beta function. We put $k = nx$, with $0 \le x \le 1$. Using the Stirling formula, $n! \sim \sqrt{2\pi}\, e^{-n}\, n^{n+1/2}$ or $\Gamma(z) \sim \sqrt{2\pi}\, e^{(z-1/2)\log z - z}$, we find

$$B(\eta\alpha+k,(1-\eta)\alpha+n-k) \sim \sqrt{\frac{2\pi}{n}}\, x^{\eta\alpha-1/2}\,(1-x)^{(1-\eta)\alpha-1/2}\, e^{-nC(x)},$$

where we introduced

$$C(x) := -x\log x - (1-x)\log(1-x),$$

with

$$C'(x) = -\log\frac{x}{1-x}, \quad C''(x) = -\frac{1}{x(1-x)}. \tag{4.10}$$

For $x \in (0,1)$, this function is non-negative, concave, and symmetric with respect to its maximum value $\log 2$ at $x = 1/2$. In fact, $C(x)$ is the basic BG (or Shannon) entropy in the case of two possibilities with probabilities x and $1-x$. Also note its relation to the rate or Cramér (also named for this particular case as Kullback-Leibler information or relative entropy) function $I(x) = \log 2 - C(x)$ appearing in the large deviation theory for the familiar fair coin-tossing model.[15] A similar function will appear in our third example ("Hermite") below. Such a relation is somewhat expected when we deal with "smooth" deformations of the binomial law. (Nevertheless, this does not apply to the Abel case where $C(x)$ is equal to 0.)

As a matter of fact, the asymptotic behavior of the binomial coefficient at large n is

$$\binom{n}{k=nx} \sim \frac{1}{\sqrt{2\pi n x(1-x)}}\, e^{nC(x))}. \tag{4.11}$$

Therefore, the limit distribution we find is the following:

$$\mathfrak{p}_{k=nx}^{(n)}(\eta) \sim \frac{1}{n}\frac{1}{B(\eta\alpha,(1-\eta)\alpha)}\, x^{\eta\alpha-1}\,(1-x)^{(1-\eta)\alpha-1}. \tag{4.12}$$

We easily check that our estimate for sums of Eq. (3.2) is valid here for the normalization of probabilities, i.e., putting $\phi_n(k) = 1$ in Eq. (3.2). Moreover, our asymptotic formula (4.12), in the case $\eta = 1/2$ and after centering in the origin, becomes proportional to a q-Gaussian[13,14] with $q = (\alpha-4)/(\alpha-2)$. This result was recently obtained numerically by Ruiz and Tsallis.[16]

B. Symmetric distribution from modified Abel polynomials

1. Definition

We take here the specific generating function $\mathcal{N}(t)$ given by

$$\mathcal{N}(t) = e^{-\alpha W(-t/\alpha)}, \quad \alpha > 0, \tag{4.13}$$

where W is the Lambert function,[17] i.e., solving the functional equation $W(t)e^{W(t)} = t$. We first note that if $\alpha \to \infty$, then $\mathcal{N}(t) \to e^t$. The corresponding sequence is bounded by α/e and given by

$$x_n = \frac{n\alpha}{n+\alpha}\left(1 - \frac{1}{n+\alpha}\right)^{n-2}, \quad \lim_{n\to\infty} x_n = \alpha/e. \tag{4.14}$$

We also note that $x_n \to n$ as $\alpha \to \infty$. The corresponding factorial is

$$x_n! = n!\,\frac{\alpha^{n-1}}{(n+\alpha)^{n-1}}. \tag{4.15}$$

The polynomials q_n's read as

$$q_n(\eta) = \eta\frac{\left(\eta + \frac{n}{\alpha}\right)^{n-1}}{\left(1 + \frac{n}{\alpha}\right)^{n-1}}. \tag{4.16}$$

We verify that $q_0(\eta) = 1$ and $q_1(\eta) = \eta$. The polynomials above are a modified version of Abel polynomials.[8] The latter are defined as

$$P_n(x) = x(x+na)^{n-1}, \; a \in \mathbb{Q}. \tag{4.17}$$

The difference between polynomials (4.17) and (4.16) is due to the presence of a normalization factor in the denominator of the latter and the relaxing of the rational condition on a.

The corresponding probability distribution is found to be

$$\mathfrak{p}_k^{(n)}(\eta) = \binom{n}{k}\eta(1-\eta)\frac{(\eta + k/\alpha)^{k-1}(1-\eta+(n-k)/\alpha)^{n-k-1}}{(1+n/\alpha)^{n-1}}, \tag{4.18}$$

with $0 \le \eta \le 1$.

2. Regularization at the limit $n \to \infty$

By putting $k = nx$ in (4.18), with $0 \le x \le 1$, and using the Stirling formula with the assumption that $nx \gg 1$ and $n(1-x) \gg 1$ while keeping n constant, we find the limit distribution

$$\mathfrak{p}_{nx}^{(n)}(\eta) \underset{\text{large } n}{\sim} \frac{\alpha\eta(1-\eta)}{\sqrt{2\pi}}(nx(1-x))^{-3/2} \equiv \rho_n(x). \tag{4.19}$$

In Figure 1 are shown the graphs of $\mathfrak{p}_{nx}^{(n)}(1/2)/\mathfrak{p}_{n/2}^{(n)}(1/2)$ in function of x for the analytic asymptotical expression, Eq. (4.19), and discrete representations of Eq. (4.18) for $\alpha = 20$ and $n = 5000$, 10 000, and 20 000. We can see in the inset that the discrete representations tend to the analytic asymptotic curve when n increases.

As mentioned in Section III, the calculations at large n of the expectation values $\langle\phi_n(k)\rangle_n$ by following the same approach as in Eq. (3.2) need some refinement in the present case, because of the divergence of the asymptotic expression $\rho_n(x)$ at the endpoints $x = 0$ and $x = 1$. This refinement is based on a regularization involving two parameters $\epsilon_1(n)$ and $\epsilon_2(n)$ introduced in Eq. (3.3). These parameters are calculated in the Appendix and the estimate of Eq. (3.3) is successfully checked for different functions $\phi_n(k)$.

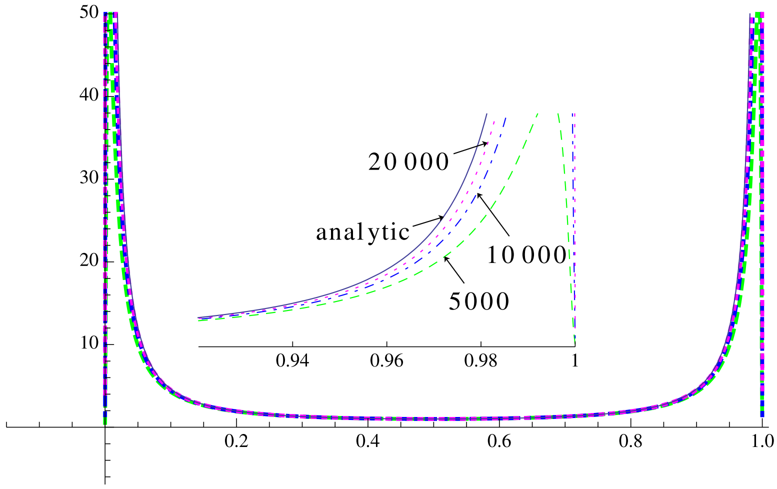


FIG. 1. Plot of $\mathfrak{p}_{nx}^{(n)}(1/2)/\mathfrak{p}_{n/2}^{(n)}(1/2)$ in function of x for $\alpha = 20$ and $\eta = 1/2$. The continuous line (fuzzy blue) is the asymptotic analytic expression, Eq. (4.19), the dashed line (green) is the curve given by Eq. (4.18) for $n = 5000$, the dotted-dashed line (blue) is for $n = 10\,000$, and the dotted line (pink) is for $n = 20\,000$. The inset is a zoom on the larger figure from 0.92 to 1; note that when n increases, the curves approach the asymptotic analytic expression given by Eq. (4.19).

3. Probabilistic interpretation

Choosing the parameters α and η in expression (4.18) as

$$\alpha = \frac{p+q}{c} \quad \text{and} \quad \eta = \frac{p}{p+q}, \tag{4.20}$$

where p, q, and c are three positive integers, we obtain

$$\mathfrak{p}_k^{(n)} = \binom{n}{k} \frac{p(p+kc)^{k-1} q(q+(n-k)c)^{n-k-1}}{(p+q)(p+q+nc)^{n-1}}. \tag{4.21}$$

From the sum of probabilities, we deduce the finite expansion formula

$$\sum_{k=0}^{n} \binom{n}{k} p(p+kc)^{k-1} q(q+(n-k)c)^{n-k-1} = (p+q)(p+q+nc)^{n-1}. \tag{4.22}$$

We now present a counting interpretation of this expansion and its resulting urn model. We define a finite set for which the numbers $\binom{n}{k} p(p+kc)^{k-1} q(q+(n-k)c)^{n-k-1}$ for $k = 0, 1 \ldots, n$ correspond to counting of partitions. As our main interest is to present at least one sound probabilistic model, for the sake of simplicity, we consider the case $c = 1$.

The model. Let $\mathcal{A}(2n, p, q) = \mathcal{A}_{C\ell}(2n) \cup \mathcal{A}_{\ell}(p) \cup \mathcal{A}_C(q)$ be an alphabet of $2n + p + q$ letters viewed as the union of three sub-alphabets:

- $\mathcal{A}_\ell(p)$, $p \geq 1$, is a set $\{b_1, b_2, \ldots, b_p\}$ of p letters which are only lowercase, by convention $\mathcal{A}_\ell(0) = \emptyset$.
- $\mathcal{A}_C(q)$, $q \geq 1$, is a set $\{C_1, C_2, \ldots, C_q\}$ of q letters which are solely capital, by convention $\mathcal{A}_C(0) = \emptyset$.
- The family $\{\mathcal{A}_{C\ell}(2n)\}_{n=1}^{\infty}$, where $\mathcal{A}_{C\ell}(2n) = \bigcup_{i=1}^{n} \{a_i, A_i\}$ made of $2n$ mixed letters, built from a possible infinite sequence of pairs

$$(a_1, A_1), \ldots (a_i, A_i), \ldots.$$

Each pair (a_i, A_i) is made from the same letter in both sizes (lowercase and capital), and the letters are assumed to be different in different pairs, independent of their size. The inclusion $\mathcal{A}_{C\ell}(2n) \subset \mathcal{A}_{C\ell}(2m)$ holds for any $n \leq m$.

In the following, we introduce also the lowercase part of $\mathcal{A}_{C\ell}(2n)$ as $\mathcal{A}_{C\ell}^{\ell}(n) = \bigcup_{i=1}^{n} \{a_i\}$ and the capital part of $\mathcal{A}_{C\ell}(2n)$ as $\mathcal{A}_{C\ell}^{C}(n) = \bigcup_{i=1}^{n} \{A_i\}$.

- All letters, independent of their size, are assumed to be different: in $\mathcal{A}(2n,p,q)$, we have $n+p$ different lowercase letters and $n+q$ different capital letters.

We consider the set of words $\mathcal{W}_n$, with n letters picked from $\mathcal{A}(2n,p,q)$, built as $\mathcal{W}_n = \bigcup_{k=0}^{n} \mathcal{W}_k^n$, where the subsets $\mathcal{W}_k^n$ contain the words with n letters, k of them being lowercase and $n-k$ capital. The words are built with the following rules:

(i) Different orderings of letters are assumed to give different words.
(ii) In a word in $\mathcal{W}_k^n$, starting from the left, the first lowercase letter encountered (if $k \neq 0$) belongs to $\mathcal{A}_\ell(p)$, and the first capital letter encountered (if $k \neq n$) belongs to $\mathcal{A}_C(q)$.
(iii) In a word in $\mathcal{W}_k^n$, all the lowercase letters ($k \neq 0$) belong to $\mathcal{A}_\ell(p) \cup \mathcal{A}_{C\ell}^{\ell}(k)$, and all the capital letters ($k \neq n$) belong to $\mathcal{A}_C(q) \cup \mathcal{A}_{C\ell}^{C}(n-k)$.

Now let us evaluate the number of words $\mathcal{N}_k^n$ in $\mathcal{W}_k^n$.

- If $k=0$, the words contain exactly n capital letters. The first one (from the left) belongs to $\mathcal{A}_C(q)$ and the $n-1$ remaining ones belong to $\mathcal{A}_C(q) \cup \mathcal{A}_{C\ell}^{C}(n)$. This gives
$$\mathcal{N}_0^n = q(q+n)^{n-1}. \tag{4.23}$$
- If $k=1$, the words contain a unique lowercase letter that belongs to $\mathcal{A}_\ell(p)$ and $n-1$ capital letters. The first capital letter belongs to $\mathcal{A}_C(q)$, the $n-2$ remaining (capital letters) belong to $\mathcal{A}_C(q) \cup \mathcal{A}_{C\ell}^{C}(n-1)$. Since there is $n = \binom{n}{1}$ ways to locate the lowercase letter in the word, we have
$$\mathcal{N}_1^n = \binom{n}{1} pq(q+n-1)^{n-2}. \tag{4.24}$$
- For $2 \le k \le n-2$, we first choose the k positions of the lowercase letters in the word, there are $\binom{n}{k}$ possibilities. The first lowercase letter belongs to $\mathcal{A}_\ell(p)$, the following $k-1$ ones belong to $\mathcal{A}_\ell(p) \cup \mathcal{A}_{C\ell}^{\ell}(k)$, then for each choice of the k positions, we have $p(p+k)^{k-1}$ possibilities for the lowercase letters. For the capital letters, we obtain similarly $q(q+n-k)^{n-k-1}$ possibilities. We deduce
$$\mathcal{N}_k^n = \binom{n}{k} p(p+k)^{k-1} q(q+n-k)^{n-k-1}. \tag{4.25}$$
- The cases $k=n-1$ and $k=n$ are analyzed by following the same rules, leading to
$$\mathcal{N}_{n-1}^n = \binom{n}{n-1} pq(p+n-1)^{n-2} \quad \text{and} \quad \mathcal{N}_n^n = p(p+n)^{n-1}. \tag{4.26}$$

We conclude that the formula of Eq. (4.25) is valid for $k=0,1,\ldots,n-1,n$. Using Eq. (4.22), we deduce that the total number $\mathcal{N}_n$ of words of $\mathcal{W}_n$ is $\mathcal{N}_n = (p+q)(p+q+n)^{n-1}$.

Remark. The value of $\mathcal{N}_n$ can be easily understood. A generic word of $\mathcal{W}_n$ contains the following:

- One letter that belongs either to $\mathcal{A}_\ell(p)$ or to $\mathcal{A}_C(q)$: this gives $p+q$ possibilities.
- Each remaining letter is either lowercase belonging to $\mathcal{A}_\ell(p) \cup \mathcal{A}_{C\ell}^{\ell}(k)$ or capital belonging to $\mathcal{A}_C(q) \cup \mathcal{A}_{C\ell}^{C}(n-k)$ for some k. This gives $(p+k)+(q+n-k) = p+q+n$ possibilities for each $n-1$ letter.

Therefore, $\mathcal{N}_n = (p+q)(p+q+n)^{n-1}$.

Conclusion. The probabilities $\mathrm{p}_k^{(n)}$ of Eq. (4.21) are the probabilities to extract a word with k lowercase letters after a draw at random from the "urn" $\mathcal{W}_n$.

Remark. Other interesting probabilities emerge from this urn model. For example, let us call $P(\{l_1,l_2,\ldots\})$ the probability that a word of $\mathcal{W}_n$ contains at least one of the letters of the family $\{l_1,l_2,\ldots\}$. We have the following results:

$$\begin{cases} P(\mathcal{A}_\ell(p)) = 1 - \mathfrak{p}_0^{(n)} \\ P(\mathcal{A}_C(q)) = 1 - \mathfrak{p}_n^{(n)} \\ \forall k \geq 2\,,\ P(\mathcal{A}_\ell(p) \cup \mathcal{A}_{C\ell}^{\ell}(k)) = \sum_{i=k}^{n} \mathfrak{p}_i^{(n)} \\ P(\mathcal{A}_\ell(p) \cup \mathcal{A}_{C\ell}^{\ell}(1)) = \frac{1}{2} P(\mathcal{A}_\ell(p) \cup \mathcal{A}_{C\ell}^{\ell}(2)) \end{cases}. \tag{4.27}$$

An example. Let us illustrate the above counting with the manageable although not trivial case $n = 3$, $p = q = 1$, and the alphabet

$$\mathcal{A} = \{a, b, c, d, A, B, C, D\} \equiv \mathcal{A}_{C\ell}(6) \cup \mathcal{A}_\ell(1) \cup \mathcal{A}_C(1),$$
$$\mathcal{A}_{C\ell}(2) = \{a, A\},\ \mathcal{A}_{C\ell}(4) = \{a, A\} \cup \{b, B\},\ \mathcal{A}_{C\ell}(6) = \{a, A\} \cup \{b, B\} \cup \{c, C\},$$
$$\mathcal{A}_{C\ell}^{\ell}(1) = \{a\},\ \mathcal{A}_{C\ell}^{\ell}(2) = \{a, b\},\ \mathcal{A}_{C\ell}^{\ell}(3) = \{a, b, c\},$$
$$\mathcal{A}_{C\ell}^{C}(1) = \{A\},\ \mathcal{A}_{C\ell}^{C}(2) = \{A, B\},\ \mathcal{A}_{C\ell}^{C}(3) = \{A, B, C\},$$
$$\mathcal{A}_\ell(1) = \{d\}, \quad \mathcal{A}_C(1) = \{D\}.$$

The total number of possible words of $\mathcal{W}_3$ is $\mathcal{N}_3 = 50$. The set of allowed words with 3 letters built from the above rules is described as follows:

- The subset of words $\mathcal{W}_0^3$ is

$$\begin{pmatrix} DAA & DAB & DAC & DAD \\ DBA & DBB & DBC & DBD \\ DCA & DCB & DCC & DCD \\ DDA & DDB & DDC & DDD \end{pmatrix},$$

  corresponding to $\mathcal{N}_0^3 = 16$ words.
- The subset of words $\mathcal{W}_1^3$ is

$$\begin{pmatrix} dDA & dDB & dDD \\ DdA & DdB & DdD \\ DAd & DBd & DDd \end{pmatrix},$$

  corresponding to $\mathcal{N}_1^3 = 9$ words.
- The subset of words $\mathcal{W}_2^3$ is

$$\begin{pmatrix} dDa & dDb & dDd \\ daD & dbD & ddD \\ Dda & Ddb & Ddd \end{pmatrix},$$

  corresponding to $\mathcal{N}_2^3 = 9$ words.
- The subset of words $\mathcal{W}_3^3$ is

$$\begin{pmatrix} daa & dab & dac & dad \\ dba & dbb & dbc & dbd \\ dca & dcb & dcc & dcd \\ dda & ddb & ddc & ddd \end{pmatrix},$$

  corresponding to $\mathcal{N}_3^3 = 16$ words.

The total number of words is $2 \times 16 + 2 \times 9 = 50$. Finally, the probabilities $\mathfrak{p}_k^{(3)}$ corresponding to these 4 situations are given in Table I.

a. Another example. Another example can be found in Genetics. First, we need to translate our set of words into the corresponding language:[19] *The nucleotide sequence of a gene (the genetic code) is a (linear) structure made of "letters" called "codons." Additionally, a "start codon" and three "stop codons" indicate the beginning and end of the protein coding region.*

TABLE I. Values of $\mathfrak{p}_k^{(n)}$ for $n=3$, $\alpha=2=p+q$, $\eta=p/(p+q)=1/2$, $p=q=1$.

k	$\mathfrak{p}_k^{(n)}$
0	8/25
1	9/50
2	9/50
3	8/25

We can use this definition to interpret our set of words as a set of nucleotide sequences. Furthermore, if we assume the three "stop codons" at the end of each sequence to be always the same, they can be omitted in our reasoning.

So, we are interested into sets $\mathcal{W}_n$ of nucleotide sequences made of n effective codons (including the start codon), the three last fixed "stop codons" being ignored. Moreover, we assume that the n effective codons possess a special chemical (or biological) property that allows to classify them as being "lowercase" or "capital."

The sets $\mathcal{W}_k^n$ are the set of sequences of n codons, k of them being "lowercase." The possible "start codons" belong to two possible sets: $\mathcal{A}_\ell(p)$ or $\mathcal{A}_C(q)$, while the remainder of codons belong to $\mathcal{A}_\ell(p)\cup\mathcal{A}_C(q)\cup\mathcal{A}_{C\ell}(2n)$. The last rule specifies that in $\mathcal{W}_k^n$, the "lowercase" codons only belong to $\mathcal{A}_\ell(p)\cup\mathcal{A}_{C\ell}^{\ell}(k)$ and the "capital" codons belong to $\mathcal{A}_C(q)\cup\mathcal{A}_{C\ell}^{C}(n-k)$. The possible reasons for such kind of structural properties are of course not specified.

Note that this model is consistent only if the role played by a codon depends on its ordering in the sequence. Actually, nucleotides used as start codons are possibly present in the sequence (in this model). So, we must assume that these codons are interpreted as the beginning of a sequence only if they are not preceded by another one of the family.

C. Symmetric distribution from Hermite polynomials

1. Definition

Here, the function $\mathcal{N}(t)$ is chosen as

$$\mathcal{N}(t)=\exp\left(t+\frac{a}{2}t^2\right),\quad 0<a<1. \tag{4.28}$$

The corresponding sequence x_n has the following factorial form:

$$\begin{aligned} x_n! &= \left[\frac{i^n\left(\frac{a}{2}\right)^{n/2}}{n!}H_n\left(\frac{-i}{\sqrt{2a}}\right)\right]^{-1} \\ &= \left[\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\frac{(a/2)^m}{m!(n-2m)!}\right]^{-1} := \frac{1}{\varphi_n(a)}. \end{aligned} \tag{4.29}$$

In particular, $x_1!=x_1=1$, $x_2!=2/(a+1)$. Also, $x_n=\varphi_{n-1}(a)/\varphi_n(a)$, and we know from Ref. 2 that $x_n\approx\sqrt{n/a}$ as $n\to\infty$. The corresponding polynomials and probability distributions are, respectively, given by

$$q_n(\eta)=\frac{x_n!}{n!}\left(i\sqrt{\frac{a\eta}{2}}\right)^n H_n\left(-i\sqrt{\frac{\eta}{2a}}\right) \tag{4.30}$$

and

$$\mathfrak{p}_k^{(n)}(\eta)=\eta^k(1-\eta)^{n-k}\frac{\varphi_k(a/\eta)\varphi_{n-k}(a/(1-\eta))}{\varphi_n(a)}. \tag{4.31}$$

2. Asymptotic behavior at large n

We now evaluate the asymptotic behavior of probability distribution (4.31). For that, let us rewrite it in terms of Hermite polynomials,

$$\mathfrak{p}_k^{(n)}(\eta) = \binom{n}{k} \eta^{\frac{k}{2}}(1-\eta)^{\frac{n-k}{2}} \frac{H_k\left(-i\sqrt{\frac{\eta}{2a}}\right) H_{n-k}\left(-i\sqrt{\frac{1-\eta}{2a}}\right)}{H_n\left(-i\sqrt{\frac{1}{2a}}\right)}. \tag{4.32}$$

Putting $k = nx$, with $0 < x < 1$, using the Stirling formula, and the asymptotic behavior of Hermite polynomials versus their respective degree when the argument is not real[18] (page 255; actually, a factor 2 in front of $|H_n(t)|$ is missing there),

$$|H_n(t)| \sim \frac{n!}{2\,\Gamma\left(\frac{n}{2}+1\right)}\, e^{\sqrt{2n}|\mathrm{Im}(t)|}, \tag{4.33}$$

we find

$$\begin{aligned}\mathfrak{p}_{k=nx}^{(n)} &\sim \frac{1}{2}\binom{\frac{n}{2}}{\frac{nx}{2}} \eta^{\frac{k}{2}}(1-\eta)^{\frac{n-k}{2}} \exp\left[\sqrt{\frac{n}{a}}(\sqrt{x\eta}+\sqrt{(1-x)(1-\eta)}-1)\right]\\ &\sim \frac{1}{2}\frac{1}{\sqrt{n\pi x(1-x)}} e^{nA(x)} \equiv \rho_n(x),\end{aligned} \tag{4.34}$$

where

$$A(x) = \frac{x}{2}\log\frac{\eta}{x} + \frac{(1-x)}{2}\log\frac{1-\eta}{1-x} + \frac{1}{\sqrt{na}}(\sqrt{x\eta}+\sqrt{(1-x)(1-\eta)}-1). \tag{4.35}$$

Using Laplace's method, it is easily checked that introducing asymptotic distribution (4.34) in our estimate of Eq. (3.2) with $\phi_n(k) = 1$ leads to the correct normalization, that is,

$$\int_0^1 \rho_n(x)\, n\mathrm{d}x = \frac{1}{2}\sqrt{\frac{n}{\pi}} \int_0^1 [x(1-x)]^{-1/2} e^{nA(x)}\, \mathrm{d}x \sim 1. \tag{4.36}$$

V. THE ENTROPIES

A. Entropies for the symmetric distributions from "q-exponential"

1. Boltzmann-Gibbs entropy

From the Boltzmann-Gibbs microscopic definition of entropy, given by (1.2), let us now establish its analytic formula in the asymptotic limit as $n \to \infty$. We use the estimate of Eq. (3.2) with $\phi_n(k) = \log\left(\mathfrak{p}_k^{(n)}/\binom{n}{k}\right) = \log \varpi_k^{(n)}$. In the present generalized distribution, which has the q-exponential as generating function, it behaves as

$$S_{\mathrm{BG}} \underset{\text{at large } n}{\sim} nI_1 + \frac{1}{2}\log n + I_2, \tag{5.1}$$

with

$$I_1 = \frac{1}{B(\eta\alpha,(1-\eta)\alpha)} \int_0^1 \mathrm{d}x\, x^{\eta\alpha-1}(1-x)^{(1-\eta)\alpha-1} C(x), \tag{5.2}$$

$$\begin{aligned}I_2 = \log\left(\frac{B(\eta\alpha,(1-\eta)\alpha)}{\sqrt{2\pi}}\right) - \frac{1}{B(\eta\alpha,(1-\eta)\alpha)}\int_0^1 \mathrm{d}x\\ x^{\eta\alpha-1}(1-x)^{(1-\eta)\alpha-1}\left[\left(\eta\alpha-\frac{1}{2}\right)\log x + \left((1-\eta)\alpha-\frac{1}{2}\right)\log(1-x)\right].\end{aligned} \tag{5.3}$$

Let us calculate the integrals appearing in the above expressions. They are all of the type

$$LB(p,q) := \int_0^1 \mathrm{d}x\, x^{p-1}\,(\log x)\,(1-x)^{q-1}$$
$$= \int_0^1 \mathrm{d}x\,(1-x)^{p-1}\,(\log(1-x))\,x^{q-1}$$
$$= \frac{\partial}{\partial p} B(p,q) = [\psi(p) - \psi(p+q)]\, B(p,q), \tag{5.4}$$

where $\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}$ is the digamma function.[18] Finally, we find that

$$S_{\mathrm{BG}} \sim n\,[\psi(\alpha+1) - (\eta\psi(\eta\alpha+1) + (1-\eta)\psi((1-\eta)\alpha+1)] + \frac{1}{2}\log n +$$
$$+ \log\left(\frac{B(\eta\alpha,(1-\eta)\alpha)}{\sqrt{2\pi}}\right)$$
$$+ \alpha\psi(\alpha) - \left[\left(\eta\alpha - \frac{1}{2}\right)\psi(\eta\alpha) + \left((1-\eta)\alpha - \frac{1}{2}\right)\psi((1-\eta)\alpha)\right]$$
$$\sim n\,[\psi(\alpha+1) - (\eta\psi(\eta\alpha+1) + (1-\eta)\psi((1-\eta)\alpha+1)]. \tag{5.5}$$

Being the value of positive integral (5.2), the slope of this linear behavior versus n is positive for any $\alpha > 0$ and $\eta \in [0,1]$. Hence, the Boltzmann-Gibbs entropy is proved to be extensive in the present case for any $\alpha > 0$ and $\eta \in [0,1]$.

2. Rényi Entropy

We now explore, for the present case, the Rényi entropy, given by (1.3). Using the asymptotic formula for the binomial coefficient at large n, (4.11) and (4.12), the approximation $\sum_{k=0}^{n} \sim \int_0^1 n\mathrm{d}x$, and the Laplace formula (see (5.25)), we obtain the asymptotic expression for $q < 1$,

$$\sum_{k=0}^{n} \binom{n}{k} \left(\varpi_k^{(n)}\right)^q \sim \frac{1}{2\sqrt{1-q}} \left[\frac{2^{5-2\alpha}\pi}{nB^2(\eta\alpha,(1-\eta)\alpha)}\right]^{q/2} e^{n(1-q)\log 2}. \tag{5.6}$$

By taking the logarithm of (5.6), we see that the dominant term is

$$S_{\mathrm{Re};q} \sim n \log 2, \tag{5.7}$$

and the Rényi entropy is obviously extensive. A point to be noticed is that this asymptotic behavior is independent of the Rényi parameter q. Actually, this remarkable feature is encountered in many distributions,[20] including the next two cases considered in this paper. We will give a special attention to this fact in the Conclusion.

B. Entropies for the symmetric distribution from modified Abel polynomials

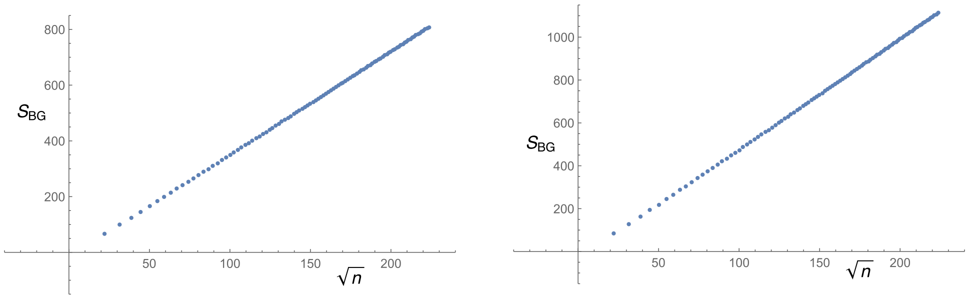
1. Boltzmann-Gibbs entropy

We now examine the Boltzmann-Gibbs entropy for distribution (4.18). We start with the numerical study of the behavior of finite sum (1.2) versus $\sqrt{n}$ as shown in Figures 2 and 3. Linear behavior (1.2) versus $\sqrt{n}$ at large n, say $n \sim 5\times 10^4$, is clearly apparent and the slope tends asymptotically to its theoretical value given below. In order to mathematically validate this result, we turn our attention to the analytic expressions derived from regularized estimate (3.3), where the regularization parameters are those specified in the Appendix and where $\rho_n(x)$ is given by Eq. (4.19), the functions ϕ_n being $\phi_n(k) = \log\left(\mathrm{p}_k^{(n)}/\binom{n}{k}\right) = \log \varpi_k^{(n)}$ (or more precisely their asymptotic expression). From (4.11), $\phi_n(nx)$ behaves asymptotically as

$$\phi_n(nx) \sim \log\left[\alpha\eta(1-\eta)(nx(1-x))^{-1} e^{-nC(x)}\right]. \tag{5.8}$$

Using Eq. (3.3), the BG entropy behaves as the sum of three terms

$$S_{\mathrm{BG}} \underset{\text{large } n}{\sim} \frac{1}{\sqrt{n}}(R_{1;n} + R_{2;n}) + \sqrt{n}R_3, \tag{5.9}$$



(a) (b)

FIG. 2. Asymptotic behavior versus $\sqrt{n}$ of the Boltzmann-Gibbs entropy for the distribution (4.18) ("Abel" case), with $n \le 50\,000$, $\alpha = 3$, and $\eta = 0.5$ (top), $\alpha = 5$, and $\eta = 0.3$ (bottom). Clearly, it behaves as $\sqrt{n}$, with a slope which tends to the theoretical value $2\sqrt{2\pi}\alpha\eta(1-\eta) \approx 3.76$ (top) ≈ 5.264 (bottom).

where

$$R_{1;n} = -\frac{\alpha\eta(1-\eta)}{\sqrt{2\pi}} \log\left(\frac{\alpha\eta(1-\eta)}{n}\right) B_n,$$
$$R_{2;n} = 2\frac{\alpha\eta(1-\eta)}{\sqrt{2\pi}} LB_n,$$
$$R_3 = -2\frac{\alpha\eta(1-\eta)}{\sqrt{2\pi}} \int_0^1 x^{-1/2}(\log x)(1-x)^{-3/2}\,\mathrm{d}x.$$

Here, we introduce the notations

$$B_n = \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{\mathrm{d}x}{(x(1-x))^{3/2}}, \qquad LB_n = \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{\log x\,\mathrm{d}x}{(x(1-x))^{3/2}}. \tag{5.10}$$

We find

$$B_n = \frac{2-4\epsilon_1}{\sqrt{\epsilon_1(1-\epsilon_1)}} + \frac{2-4\epsilon_2}{\sqrt{\epsilon_2(1-\epsilon_2)}}, \qquad LB_n = F(1-\epsilon_2) - F(\epsilon_1), \tag{5.11}$$

with

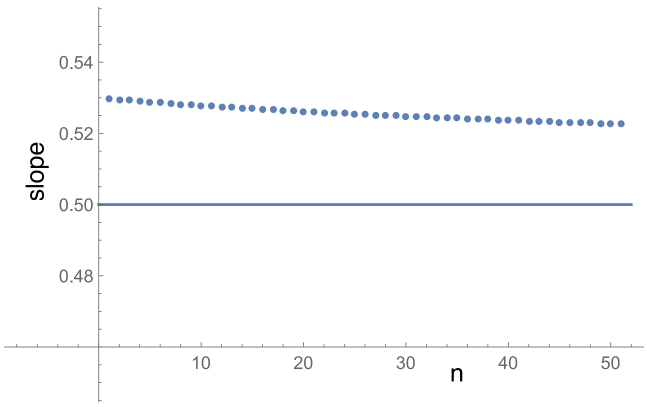


FIG. 3. Behavior versus n of the slope of the log-log plot of the Boltzmann-Gibbs entropy versus n for distribution (4.18) ("Abel" case), with $n \le 50\,000$, $\alpha = 5$, and $\eta = 0.3$ (top). It tends (slowly) to the theoretical value 1/2 given by (5.18).

$$F(a) = 12 \arcsin \sqrt{1-a} - \frac{2-4a}{\sqrt{a(1-a)}} \log a - \frac{4(1-a)}{\sqrt{a(1-a)}}. \tag{5.12}$$

Taking into account the expressions of ϵ_1 and ϵ_2 given in Eqs. (A9) and (A8), we obtain

$$B_n \sim \frac{\sqrt{2\pi n}}{\alpha\eta(1-\eta)} \quad \text{and} \quad LB_n \sim \frac{\sqrt{2\pi n}}{\alpha\eta} \log n. \tag{5.13}$$

Therefore, we obtain

$$R_{1;n} \sim \sqrt{n} \log n \quad \text{and} \quad R_{2;n} \sim 2(1-\eta)\sqrt{n} \log n. \tag{5.14}$$

The value of R_3 is easily found from Ref. 21,

$$LI_p := \int_0^1 x^{p-1} (\log x) (1-x)^{-p-1} \, \mathrm{d}x = -\frac{\pi}{p} \csc p\pi, \quad 0 < p < 1. \tag{5.15}$$

In the present case, $p = 1/2$ and so $LI_p = -2\pi$ and

$$R_3 = 2\sqrt{2\pi}\alpha\eta(1-\eta). \tag{5.16}$$

From these results, we find that the dominant terms in (5.9) are

$$S_{\mathrm{BG}} \sim (3-2\eta) \log n + \sqrt{n}\, 2\sqrt{2\pi}\, \alpha\, \eta\, (1-\eta) \sim \sqrt{n}\, 2\sqrt{2\pi}\, \alpha\, \eta\, (1-\eta), \tag{5.17}$$

and finally,

$$S_{\mathrm{BG}} \sim \sqrt{n}\, 2\sqrt{2\pi}\, \alpha\, \eta\, (1-\eta). \tag{5.18}$$

We conclude that the Boltzmann-Gibbs entropy *is not extensive* for this type of deformation of the binomial distribution. It behaves as $\sqrt{n}$ with slope equal to $2\sqrt{2\pi}\, \alpha\, \eta\, (1-\eta)$, in agreement with the numerical calculations shown in Figures 2 and 3. This surprising result shows that the extensivity of the Boltzmann-Gibbs entropy can change depending on the correlation among the events, a theoretical question debated in the literature for many years.[22–24] Here, we have a confirmation of that.

2. Rényi Entropy

Using the estimate of Eq. (3.2) in (1.3) (under the condition $q < 1$) followed by a Laplace approximation yields

$$\sum_{k=0}^{n} \binom{n}{k} \left(\varpi_k^{(n)}\right)^q \sim \frac{1}{\sqrt{|q-1|}}\, 2^{2q} (\alpha\eta(1-\eta))^q\, n^{-q}\, e^{n(1-q)\log 2}, \tag{5.19}$$

and we derive immediately

$$\log \left[\sum_{k=0}^{n} \binom{n}{k} \left(\varpi_k^{(n)}\right)^q \right] \sim \log \left(\frac{1}{\sqrt{|q-1|}}\, 2^{2q} (\alpha\eta(1-\eta))^q \right) + \\ - q \log n + n(1-q) \log 2. \tag{5.20}$$

Therefore, the Rényi entropy is extensive for $q < 1$,

$$S_{\mathrm{Re};q} \sim n\, \log 2. \tag{5.21}$$

We recover the asymptotic q-independence already noticed in the "q-exponential" case.

C. Entropies for the symmetric distribution from Hermite polynomials

1. Boltzmann-Gibbs entropy

From asymptotic behavior (4.34) and (4.11), we infer the following behavior:

$$\binom{n}{nx} \varpi_{nx}^{(n)} \log \varpi_{nx}^{(n)} \sim \frac{1}{2\sqrt{n\pi}}\, h(x)\, e^{nA(x)}, \tag{5.22}$$

where $A(x)$ is given by (4.35) and the function $h(x)$ is given by

$$h(x) = \sqrt{x(1-x)}\left[-\frac{1}{2}\log 2 + \frac{n}{2}[x\log(x\eta) + (1-x)\log((1-x)(1-\eta))]+ \right. \\ \left. + \sqrt{\frac{n}{a}}[\sqrt{x\eta} + \sqrt{(1-x-(1-\eta)} - 1]\right]. \tag{5.23}$$

After the usual replacement $\sum_{k=0}^{n} \mapsto \int_0^1 n\mathrm{d}x$, we get for BG entropy (1.2),

$$S_{\mathrm{BG}} \sim -\frac{1}{2}\sqrt{\frac{n}{\pi}}\int_0^1 h(x)\, e^{nA(x)}\,\mathrm{d}x. \tag{5.24}$$

Applying the Laplace approximation method,

$$S_{\mathrm{BG}} \sim \frac{1}{2}\log 2 - n[\eta\log\eta + (1-\eta)\log(1-\eta)] \sim n[\eta\log(1/\eta) + (1-\eta)\log(1/(1-\eta))].$$

So, we can conclude that S_{BG} is extensive in this model.

2. Rényi entropy

To estimate the asymptotic behavior of the Rényi entropy, we first use the approximation resulting from (4.34) and (4.11),

$$\left[\binom{n}{k=nx}\right]^{1-q}\left(\mathrm{p}_{k=nx}^{(n)}\right)^q \sim \frac{1}{\sqrt{2^{q+1}n\pi x(1-x)}}e^{nB(x)}, \tag{5.25}$$

with

$$\begin{aligned} B(x) = qA(x) - (q-1)C(x) = \frac{q}{2}[x\log\eta + (1-x)\log(1-\eta)]+ \\ + \left(\frac{q}{2}-1\right)[x\log x + (1-x)\log(1-x)]+ \\ + \frac{q}{\sqrt{na}}[\sqrt{x\eta} + \sqrt{(1-x)(1-\eta)} - 1]. \end{aligned} \tag{5.26}$$

Next, we transform the sum into an integral, as usual,

$$\sum_{k=0}^{n}\left[\binom{n}{k}\right]^{1-q}\left(\mathrm{p}_k^{(n)}\right)^q \sim \sqrt{\frac{n}{2^{q+1}\pi}}\int_0^1 \mathrm{d}x\,(x(1-x))^{-1/2}\,e^{nB(x)}. \tag{5.27}$$

In order to implement the Laplace method, we calculate B' and B'',

$$B'(x) = \frac{q}{2}[\log\eta - \log(1-\eta)] + \left(\frac{q}{2}-1\right)[\log x - (1-x)\log(1-x)]+ \\ + \frac{q}{2\sqrt{na}}\left[\sqrt{\frac{\eta}{x}} + \sqrt{\frac{1-\eta}{1-x}}\right], \tag{5.28}$$

$$B''(x) = \left(\frac{q}{2}-1\right)\frac{1}{x(1-x)} - \frac{q}{4\sqrt{na}}\left[\sqrt{\frac{\eta}{x^3}} + \sqrt{\frac{1-\eta}{(1-x)^3}}\right]. \tag{5.29}$$

We see that for $q < 2$, we have $B''(x) < 0$ for all $x \in (0,1)$. Hence, if $q < 2$ and if we find one and only one $x_0 \in (0,1)$ such that $B'(x_0) = 0$, the Laplace approximation method is valid, and we obtain the behavior of the sum at large n,

$$\sum_k \left[\binom{n}{k}\right]^{1-q}\left(\mathrm{p}_k^{(n)}\right)^q \sim \sqrt{\frac{1}{2^q|B''(x_0)|}}\,(x_0(1-x_0))^{-1/2}\,e^{nB(x_0)}. \tag{5.30}$$

Now, for the median value $\eta = 1/2$, we find immediately the unique solution $x_0 = 1/2$. Then, $B''(1/2) = 2(q-2) - q/\sqrt{na}$, $B(1/2) = (1-q)\log 2$, and so,

$$\sum_k \left[\binom{n}{k}\right]^{1-q}\left(\mathrm{p}_k^{(n)}\right)^q \underset{\text{at large } n}{\sim} 2^{(3-q)/2}(q-2)^{-1/2}\,e^{n(1-q)\log 2}. \tag{5.31}$$

Therefore, for $\eta = 1/2$, the Rényi entropy is extensive,

$$S_{\mathrm{Re};q} \sim n \log 2. \tag{5.32}$$

One can easily show that with $\eta = 1/2 + \delta$, $|\delta| \ll 1/2$, the value of the root x_0 is $x_0 = \frac{q}{2-q}\delta + O(\delta^2)$ and that behavior (5.32) holds too. We have checked numerically that it holds for all $\eta \in (0,1)$. We notice that this behavior (which is simply $\sim n$ if we adopt the original Rényi choice $\log_2$) is the same as for the two other cases considered in this paper, Eqs. (5.7) and (5.21), and also for the binomial and Laplace de Finetti distributions considered in Ref. 20. We will come back to this important point in the Conclusion.

VI. CONCLUSION(S)

In this paper, our main interest is the extensivity property of different entropies constructed from generalized binomial distributions. We examine the behavior of two entropies, namely, the Boltzmann-Gibbs and Rényi ones, for the three examples of generalized binomial distributions presented in Ref. 4, based on the probabilities of the possible states. For that sake, we examined the asymptotic behavior of the deformed probability distributions in question, which are those whose generating functions are the q-exponential, the exponential of the Lambert function, and the exponential of a second-degree polynomial: the probabilities obtained are, respectively, the Pólya distribution, a product of modified Abel polynomials and a product of Hermite polynomials. The study is analytical for two examples (q-exponential and Hermite) and both numerical and analytical for the Abel case.

The results found for those two entropies are interesting on three different levels.

(i) First, the Rényi entropy is extensive for the three probability distributions. An important aspect of the result found here is that for all the three studied distributions, its asymptotic value at large n is the same, $n \log 2$, and therefore, does not depend on its parameter $q \in (0,1)$.

(ii) Second, we observe that the two limits $n \to \infty$ and $q \to 1$ do not commute in the three cases. For the q-exponential and Hermite cases, the two entropies are extensive but have different amplitudes.

(iii) Finally, and this is surprising, the Boltzmann-Gibbs one is extensive for two cases, those related to the q-exponential and to the Hermite polynomials, but not when the probability distribution is given by modified Abel polynomials. That this latter fact be related to regularization techniques is an interesting question to be examined in further investigations.

This example of non-extensivity of Boltzmann-Gibbs is a result that deserves further investigation, as it has so far been considered as the universally extensive entropy, see Refs. 20 and 22–24 for recent discussions and developments about this crucial point. Actually, this extensivity is probably due to the nature of the three distributions examined here, which are smooth deformations of the binomial one. We have shown in Ref. 20 that both Boltzmann-Gibbs and Rényi are extensive entropies for the binomial case. Deformations of the binomial distribution introduce correlations, and these correlations may or may not be strong enough to substantially modify the asymptotic behaviors. The fact that extensivity holds for Rényi and for its BG limit at $q = 1$ when the deformed probability is either the Pólya distribution or a product of Hermite polynomials indicates that in these cases, the related correlations are not strong enough to modify the extensivity property of the Boltzmann-Gibbs entropy. Otherwise, the behavior of the deformed probability given as products of modified Abel polynomials is different as the Boltzmann-Gibbs limit of the Rényi entropy is asymptotically not extensive. This distribution deserves a further investigation on the correlations it introduces and we might expect them to be stronger than the two former mentioned cases; this issue will be the subject of future work. Due to this exceptionality of the modified Abel polynomials case, we illustrated it here with a concrete and non-trivial probabilistic model.

ACKNOWLEDGMENTS

The authors thank the Referee for pertinent comments and constructive suggestions. H. Bergeron and J. P. Gazeau thank the CBPF and the CNPq for financial support and CBPF for hospitality. E. M. F. Curado acknowledges CNPq and FAPERJ for financial support.

APPENDIX: REGULARIZATION FOR THE MODIFIED ABEL POLYNOMIALS

1. Finding the regularization parameters

As mentioned in Section III, we introduce the regularization parameters $\epsilon_1(n)$ and $\epsilon_2(n)$ (vanishing for large n) that lead to the estimate of Eq. (3.3). Using the asymptotic expression of Eq. (4.19), we obtain

$$\sum_{k=0}^{n} \phi_n(k)\mathfrak{p}_k^{(n)}(\eta) \underset{n\to\infty}{\sim} \frac{\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{\mathrm{d}x}{(x(1-x))^{3/2}} \phi_n(nx). \tag{A1}$$

The normalization of probability and the expectation value $\langle k \rangle_n = n\eta$ (see Eq. (32) in Ref. 4) lead to the constraints

$$\frac{\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{\mathrm{d}x}{(x(1-x))^{3/2}} \underset{n\to\infty}{\sim} 1 \tag{A2}$$

and

$$\frac{\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{\mathrm{d}x}{(x(1-x))^{3/2}} nx \underset{n\to\infty}{\sim} n\eta. \tag{A3}$$

Equation (A2) reads

$$\frac{\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \left(\frac{2-4\epsilon_1(n)}{\sqrt{\epsilon_1(n)(1-\epsilon_1(n))}} + \frac{2-4\epsilon_2(n)}{\sqrt{\epsilon_2(n)(1-\epsilon_2(n))}} \right) \underset{n\to\infty}{\sim} 1. \tag{A4}$$

Furthermore, the integral involved in Eq. (A3) is regular at the endpoint $x = 0$. Therefore, Eq. (A3) simplifies as

$$\alpha\eta(1-\eta)\sqrt{\frac{n}{2\pi}} \int_{0}^{1-\epsilon_2(n)} \frac{\mathrm{d}x}{(x(1-x))^{3/2}} nx \underset{n\to\infty}{\sim} n\eta \tag{A5}$$

or

$$\frac{2\sqrt{n}\,\alpha\eta(1-\eta)}{\sqrt{2\pi}} \frac{\sqrt{1-\epsilon_2(n)}}{\sqrt{\epsilon_2(n)}} \underset{n\to\infty}{\sim} n\eta. \tag{A6}$$

Since $\epsilon_2(n)$ is vanishing for large n, we obtain the behavior

$$\epsilon_2(n) \underset{n\to\infty}{\sim} \frac{2\alpha^2}{\pi n}(1-\eta)^2. \tag{A7}$$

Our purpose being to obtain estimates only for large values of n, we deduce that the sought parameter $\epsilon_2(n)$ is given by

$$\epsilon_2(n) = \frac{2\alpha^2}{\pi n}(1-\eta)^2. \tag{A8}$$

Introducing this expression of $\epsilon_2(n)$ in Eq. (A4) and taking into account the vanishing character of $\epsilon_1(n)$, we obtain the following expression of $\epsilon_1(n)$:

$$\epsilon_1(n) = \frac{2\alpha^2}{\pi n}\eta^2. \tag{A9}$$

2. Checking the general formula

The regularization parameters being given by Eqs. (A9) and (A8), one can wonder if the general estimate of Eq. (A1) holds true for any C^∞ functions ϕ_n, but we are unable to prove a so strong statement.

By construction, the estimate holds true for $\phi_n(k) = 1$ and $\phi_n(k) = k$, and we can only check the formula for other particular cases as $\phi_n(k) = k^2$ and $\phi_n(k) = k(n-k)$.

Indeed, using our previous paper[4] (Eqs. (33), (36), (59), and (62)), we have

$$\langle k^2 \rangle_n = \eta n^2 + \eta(1-\eta)c_n \quad \text{and} \quad \langle k(n-k)\rangle_n = \eta(1-\eta)c_n, \tag{A10}$$

where $\langle . \rangle_n$ holds for expectation values, and with

$$c_n = \frac{\alpha n(n-1)}{(n+\alpha)^{n-1}} e^{\alpha+n}\Gamma(n-1, \alpha+n), \tag{A11}$$

$\Gamma(a,x)$ being the incomplete gamma function.

Therefore, we obtain the following asymptotic expressions for large n:

$$\langle k^2 \rangle_n \underset{n\to\infty}{\sim} \eta\, n^2 \text{ and } \langle k(n-k)\rangle_n \underset{n\to\infty}{\sim} \sqrt{\frac{\pi}{2}}\alpha\eta(1-\eta)n^{3/2}. \tag{A12}$$

a. Checking $\langle k^2 \rangle_n$

The general formula of Eq. (A1) reads

$$\sum_{k=0}^{n} k^2\, \mathfrak{p}_k^{(n)}(\eta) \underset{n\to\infty}{\sim} \frac{n^2\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{x^2 \mathrm{d}x}{(x(1-x))^{3/2}}. \tag{A13}$$

Since the integral is convergent at the endpoint $x = 0$, we can remove ϵ_1, then

$$\frac{n^2\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{x^2 \mathrm{d}x}{(x(1-x))^{3/2}} \underset{n\to\infty}{\sim} \frac{n^{3/2}\alpha\eta(1-\eta)}{\sqrt{2\pi}} \int_0^{1-\epsilon_2} \frac{\sqrt{x}\mathrm{d}x}{(1-x)^{3/2}}. \tag{A14}$$

After integration by parts, we obtain

$$\frac{n^2\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{x^2 \mathrm{d}x}{(x(1-x))^{3/2}} \underset{n\to\infty}{\sim} \frac{2n^{3/2}\alpha\eta(1-\eta)}{\sqrt{2\pi}} \frac{\sqrt{1-\epsilon_2}}{\sqrt{\epsilon_2}}. \tag{A15}$$

Taking into account the expression of ϵ_2, we get the expected result

$$\frac{n^2\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{x^2 \mathrm{d}x}{(x(1-x))^{3/2}} \underset{n\to\infty}{\sim} n^2\eta. \tag{A16}$$

b. Checking $\langle k(n-k) \rangle_n$

The general formula of Eq. (A1) reads

$$\sum_{k=0}^{n} k(n-k)\, \mathfrak{p}_k^{(n)}(\eta) \underset{n\to\infty}{\sim} \frac{n^2\,\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{x(1-x) \mathrm{d}x}{(x(1-x))^{3/2}}. \tag{A17}$$

The integral being now convergent on both ends, the regularization can be removed leading again to the expected expression

$$\frac{n^2\,\alpha\eta(1-\eta)}{\sqrt{2\pi n}} \int_{\epsilon_1(n)}^{1-\epsilon_2(n)} \frac{x(1-x) \mathrm{d}x}{(x(1-x))^{3/2}} \underset{n\to\infty}{\sim} \sqrt{\frac{\pi}{2}}\alpha\eta(1-\eta)n^{3/2}. \tag{A18}$$

[1] E. M. F. Curado, J. P. Gazeau, and L. M. C. S. Rodrigues, "On a generalization of the binomial distribution and its Poisson-like limit," J. Stat. Phys. **146**, 264-280 (2012).
[2] H. Bergeron, E. M. F. Curado, J. P. Gazeau, and L. M. C. S. Rodrigues, "Generating functions for generalized binomial distributions," J. Math. Phys. **53**, 103304-1-103304-22 (2012).
[3] H. Bergeron, E. M. F. Curado, J. P. Gazeau, and L. M. C. S. Rodrigues, "Generalized binomial distributions," in *Group 29: Physical and Mathematical Aspects of Symmetries (Proceedings of the 29th International Colloquium on Group Theoretical Methods in Physics, Tianjin, China, 25 August 2012)*, edited by J. P. Gazeau, M. Ge, and C. Bai (World Scientific, 2013), Vol. 19, pp. 265-270.
[4] H. Bergeron, E. M. F. Curado, J. P. Gazeau, and L. M. C. S. Rodrigues, "Symmetric generalized binomial distribution," J. Math. Phys. **54**, 123301-1-123301-18 (2013); e-print arXiv:1308.4863v1 [math-ph].
[5] C. E. Shannon, "A mathematical theory of communication," Bell Syst. Tech. J. **27**, 379-423 and 623-656 (1948).
[6] C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication* (University of Illinois Press, Urbana, 1949).
[7] A. Rényi, "On measures of entropy and information," in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, 1960), Vol. 1, pp. 547-561.
[8] S. Roman, *The Umbral Calculus*, Pure and Applied Mathematics Vol. 111 (Academic Press, Inc. [Harcourt Brace Jovanovich Publishers], London, 1984), reprinted by Dover, 2005, pp. 29–30 and pp. 72–75.
[9] C. Tsallis, "Nonadditive entropy and nonextensive statistical mechanics-an overview after 20 years," Braz. J. Phys. **39**, 337-356 (2009).
[10] N. Johnson, S. Kotz, and A. W. Kempf, *Univariate Discrete Distributions*, 2nd ed. (John Wiley and Sons, 1992), p. 239.
[11] See http://www.encyclopediaofmath.org/ for Pólya distribution–Encyclopedia of Mathematics.
[12] R. Hanel, S. Thurner, and C. Tsallis, "Limit distributions of scale-invariant probabilistic models of correlated random variables with the q-Gaussian as an explicit example," Eur. Phys. J. B **72**, 263-268 (2009).
[13] J. A. Marsh, M. A. Fuentes, L. G. Moyano, and C. Tsallis, "Influence of global correlations on central limit theorems and entropic extensivity," Physica A **372**, 183-202 (2006).
[14] A. Rodríguez and C. Tsallis, "Connection between Dirichlet distributions and a scale-invariant probabilistic model based on Leibniz-like pyramids," J. Stat. Mech. **2014**, P12027.
[15] S. R. S. Varadhan, "Special invited paper: Large deviations," Ann. Probab. **36**, 397-419 (2008).
[16] G. Ruiz and C. Tsallis, "Emergence of q-statistical functions in a generalized binomial distribution with strong correlations," J. Math. Phys. **56**, 053301-053312 (2015).
[17] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W function," Adv. Comput. Math. **5**, 329-359 (1996).
[18] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed. (Springer-Verlag, Berlin, 1966).
[19] See http://en.wikipedia.org/wiki/Gene for Wikipedia.
[20] H. Bergeron, E. M. F. Curado, J. P. Gazeau, and L. M. C. S. Rodrigues, "Extensivity of Rényi entropy in the Laplace-de Finetti distributions," Physica A **441**, 23-31 (2016).
[21] I. S. Gradshteyn and I. M. Ryzhik, in *Table of Integrals, Series, and Products*, 7th ed., edited by A. Jeffrey and D. Zwillinger (Academic Press, New York, 2007).
[22] R. Hanel and S. Thurner, "When do generalized entropies apply? How phase space volume determines entropy," EPL **96**, 50003 (2011).
[23] R. Hanel, S. Thurner, and M. Gell-Mann, "How multiplicity determines entropy and the derivation of the maximum entropy principle for complex systems," Proc. Natl. Acad. Sci. U. S. A. **111**, 6905-6910 (2014).
[24] C. Tsallis, "An introduction to nonadditive entropies and a thermostatistical approach to inanimate and living matter," Contemp. Phys. **55**, 179-197 (2014).