Article Title Page

[Article title] Generalized q-Weibull model and the bathtub curve

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Structured Abstract: Purpose
To analyze mathematical aspects of the q -Weibull model and explore the influence of the parameter q.
Metodology
Analytical developments with graph illustrations and an application to a practical example.

Findings
The q -Weibull distribution function is able to reproduce the bathtub shape curve for the failure rate function. Moments of the distribution are also presented.

Practical implications
The generalized q -Weibull distribution unifies various possible descriptions for the failure rate function: monotonically decreasing, monotonically increasing, unimodal and U-shaped (bathtub) curves. It

recovers the usual Weibull distribution as a particular case. It represents a unification of models usually found in reliability analysis. q -Weibull model has its inspiration in nonextensive statistics, used to describe complex systems with long-range interactions and/or long-term memory. This theoretical background may help the understanding of the underlying mechanisms for failure events in engineering problems.

Originality
q-Weibull model has already been introduced in the literature, but it was not realized that it is able to reproduce a bathtub curve. The paper brings a mapping of the parameters, showing the range of the parameters that should be used for each type of curve. It also brings moments of the distribution and other mathematical details.

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1 Introduction

Reliability analysis largely uses Weibull distribution (Weibull, 1951), that is a simple and powerful empirical model. Many branches of knowledge has applied this distribution. These are some recent examples: service operations (Hensley and Utley, 2011), the problem of the strength of a manufactured item against stress (Ali and Kannan, 2011) and large-scale information systems supporting infrastructures deterioration process formulated by a Weibull hazard model (Kobayashi and Kaito, 2011). Weibull probability density function (pdf) at time t, where $t < T$ and T is time to failure, is given by

$$f(t) = \frac{\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta-1} \exp\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right] \qquad (1)$$

with $\beta > 0$, $\eta > 0$, $\eta > t_0$, $t \geq t_0$, and $\int_0^{\infty} f(x)dx = 1$. Eq. (1) may be viewed as a generalization of the exponential distribution, that is recovered if parameter β is taken as unity.

Various generalizations of Weibull model have been proposed: linear or nonlinear transformation of time, use of multiple distributions, time dependence of parameters, discrete, multivariate, stochastic models, etc. (see (Murthy et al., 2004)) for a comprehensive approach). (Xie et al., 2000) compares the approximated exponential distribution using the average failure rate with the Weibull reliability. Almost all proposals of generalization of Weibull model share a common feature: they rely on the *exponential* framework (single exponential, exponentials of a variety of functions and so forth).

In the following we briefly point out some theoretical remarks about the emergence of exponential and non exponential distributions in statistical mechanics, that serve as motivation for our approach to the problem. Exponentials are usually found in non-interacting or weakly interacting systems. Systems that exhibit long-range (spatial) interactions, long-term (temporal) memory, effects of competition/cooperation, among others, usually can be classified as *complex* (see, for instance, (Bak, 1997)) and power-laws dominate their statistical distributions, in contrast to simple systems, that is the realm of exponential laws. Failure of a component may have many (recent or not) multiple and interacting causes, some of them acting on a cooperative and others on a conflictive basis, so it is not surprising that complex behavior may appear. If this happens, power-law-like expressions are expected to substitute exponentials in the statistical description.

Statistical mechanics of simple systems has a well established theoretical framework, and probability distributions with exponentials (e.g. Boltzmann weight, Maxwellian distribution among many others) are derived from Boltzmann-Gibbs-Shannon (BGS) entropy. On the other hand, theoretical basis of the statistical description of complex systems is object of intense current research.

The definition of the nonextensive entropy (Tsallis, 2009), that is a generalization of BGS entropy, (by means of a parameter q, also known as entropic index), has introduced the possibility to extend statistical mechanics to complex systems in a coherent and natural way. The developments surpassed the bounds of physics and have lead to applications in different areas, including topics in applied mathematics. We focus on the q-exponential function, that naturally appears in nonextensive formalism, defined as

$$\exp_q(x) = \begin{cases} \left(1+(1-q)x\right)^{\frac{1}{1-q}}, & \text{if } \left(1+(1-q)x\right) \geq 0 \\ 0, & \text{otherwise,} \end{cases} \qquad (2)$$

with $x, q \in \mathrm{R}$. The q-exponential is reduced to the usual exponential function in the limiting case $q \to 1$ ($\exp_1 x = \exp x$), and thus Eq. (2) is a generalization of the later. The definition of the q-exponential brings a cut-off condition that prevents negative or even complex values. This is an important feature whenever the function is to be associated with probabilities. For certain values of the parameters the q-exponential presents a crossover between an exponential behavior and a power-law regime ($\exp_q(-ax)$ with $a > 0$ and $q > 1$ is asymptotically a power-law for large x, leading to fat-tailed distributions).

The q-exponential has been applied to different contexts in pure and applied mathematics. For the present purposes we are particularly interested in the applications in probability distributions. The q-gaussian distribution (Tsallis et al.,1995, Prato and Tsallis, 1999) generalizes the gaussian (recovered for $q = 1$), and also the Cauchy-Lorentz distribution (recovered for $q = 2$), among others. The central limit theorem has been generalized into its "q-version" in (Tsallis, 2005) and (Umarov et al., 2008).

If we look to Weibull distribution on the light of nonextensive statistics, a natural step forward is its generalization with q-exponentials, and this was done in (Picoli et al., 2003), with applications in frequency distributions for different systems. To the best of our knowledge, the first use of q-Weibull distribution in reliability analysis was presented in (Costa et al., 2006). It was applied to describe time-to-breakdown during the dielectric breakdown regime of ultra-thin oxides in electronic devices. q-Weibull pdf was also used in order to model data of New York Stock Exchange and Helsinki Stock Exchange (Vuorenmaa, 2006).

The aim of the present paper is to recall q-Weibull model and to analyze some features and details that are important to reliability analysis and were not covered earlier. It is a continuation of a previous paper (Sartori et al., 2009), in which we have done a preliminary study of the applicability of q-Weibull distribution, and also a continuation of (Assis et al., 2011), in which we have compared q-Weibull pdf with q-exponential pdf. The present paper shows that q-Weibull distribution is able to reproduce various types of failure rate behaviors: monotonically decreasing, monotonically increasing, unimodal and U-shaped (bathtub curve). The possibility to use q-Weibull to describe the bathtub curve was not realized by previous papers. Before introducing the model (what is done in the next Section), we show Figure 1 that compares Weibull distribution and the q-Weibull distribution. Two curves of the Weibull distribution are displayed, a decreasing function (with shape parameter $\beta < 1$), and an increasing function (with shape parameter $\beta > 1$). The q-Weibull model approximates both curves, for small and large values of time, and properly interpolates in-between, generating the curve with the bathtub shape.

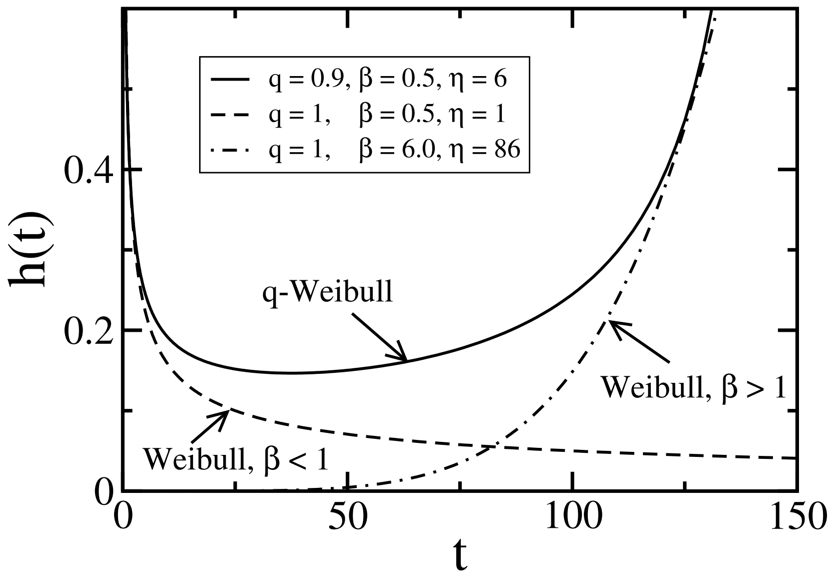

Figure 1 Comparison of two instances of the Weibull distribution, the decreasing curve with shape parameter $\beta = 0.5$, and the increasing curve with shape parameter $\beta = 6$. The displayed q-Weibull distribution is a generalization of ordinary Weibull and is able to represent the bathtub curve. The values of the parameters were chosen just to give a good visual representation.

Section 2 introduces the model and some of its features are shown in Section 3. Section 4 brings an example and our conclusions and final remarks are developed in Section 5.

2 *q*-Weibull failure rate model

The q-Weibull model is obtained from the classical Weibull model (Eq. (1)) by the substitution of the exponential function by a q-exponential (see details in (Costa et al., 2006)):

$$f_q(t) = (2-q)\frac{\beta}{\eta - t_0}\left(\frac{t-t_0}{\eta - t_0}\right)^{\beta-1} \exp_q\left[-\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]. \qquad (3)$$

The factor $(2-q)$ and the constraint $q < 2$ are necessary for normalizability requirements. The ordinary Weibull pdf is recovered in the limit $q \rightarrow 1$, and coherently Eq. (1) shall now be denoted as $f_1(t)$. η is the scale parameter and t_0 is the location parameter of q-Weibull model as well as in Weibull model, however β and q parameters control the shape of q-Weibull distribution while in Weibull model only β affects the shape.

q-Weibull is also a generalization of Burr XII distribution function (Burr, 1942),

$$f(t) = ck\frac{t^{c-1}}{s^c}\left[1+\left(\frac{t}{s}\right)^c\right]^{-k-1} \quad (k>0, c>0, s>0), \qquad (4)$$

if the parameters of q-Weibull are taken as $\beta = c$, $\eta = s/(k+1)^{1/c}$ and $q = (k+2)/(k+1) > 1$. It is worth a mention that q-Weibull is a generalization of Burr XII, and not the opposite as claimed by (Nadarajah and Kotz, 2006), once Eq. (4) demands $q > 1$, while Eq. (3) is also defined for $q \leq 1$. Burr XII distribution can assume different shapes which allow it to be a good candidate to fit various lifetimes data. Recent studies in (Rastogi and Tripathi, 2011) estimate an unknown parameter of the Burr type XII distribution when data are hybrid censored.

The q-Weibull reliability function is consistently given by

$$\begin{aligned} R_q(t) &= \int_t^\infty f_q(t')dt' \\ &= \left[1-(1-q)\left(\frac{t-t_0}{\eta-t_0}\right)^\beta\right]_+^{\frac{2-q}{1-q}} \qquad (5) \\ &= \left[\exp_q\left[-\left(\frac{t-t_0}{\eta-t_0}\right)^\beta\right]\right]^{2-q}, \end{aligned}$$

where we use the symbol $[A]_+$ (second line of Eq. 5) that means that $[A]_+ = A$ if $A \geq 0$ and $[A]_+ = 0$ if $A < 0$. This is already implicit in Eq. 2: we use it here and also in some equations in the following just to remind the reader of the cut-off condition of the q-exponential. In order to arrive at Eq. (5) we have used the following property of the q-exponential function:

$$\int \exp_q(ax)dx = \frac{1}{(2-q)a}[\exp_q(ax)]^{2-q}. \qquad (6)$$

Note that $(\exp_q x)^a \neq \exp_q(ax)$ for $q \neq 1$, but

$$(\exp_q x)^a = \exp_{1-(1-q)/a}(ax), \qquad (7)$$

so that Eq. (5) may be alternatively written as $R_q(t) = \exp_{q'}[-(2-q)((t-t_0)/(\eta-t_0))^\beta]$ with $q' = 1/(2-q)$. The interested reader may find more properties of q-exponentials at (Yamano, 2002).

The cumulative distribution function $F_q(t)$ is the complement to the reliability function,

$$F_q(t) = 1 - R_q(t). \qquad (8)$$

The instantaneous failure rate, defined as

$$h_q(t) \equiv \frac{f_q(t)}{R_q(t)} \qquad (9)$$

is generalized to

$$\begin{aligned} h_q(t) &= \frac{(2-q)\beta}{\eta - t_0}\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta-1} \times \left[1-(1-q)\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]_+^{-1} \\ &= \frac{(2-q)\beta}{\eta - t_0}\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta-1} \times \left[\exp_q\left[-\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]\right]^{q-1}, \end{aligned} \qquad (10)$$

which is consistently reduced to the usual Weibull version as $q \to 1$:

$$h_1(t) = \frac{\beta}{\eta - t_0}\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta-1}. \qquad (11)$$

This is precisely the origin of the difference of behaviors between usual ($q = 1$) and q-Weibull models: the integral of an ordinary exponential is an exponential (except from a multiplicative constant), and they cancel out in the expression for the failure rate (Eq. (9) with $q = 1$). That does not happen with $h_q(t)$, due to the property given by Eq. (6).

Equation (10) is able to represent four different types of failure rate function, according to the values of the parameters, besides the constant type (with $q = 1$ and $\beta = 1$). $h_q(t)$ is monotonically decreasing for $1 < q < 2$ and $0 < \beta < 1$, monotonically increasing for $q < 1$ and $\beta > 1$, unimodal for $1 < q < 2$ and $\beta > 1$ and U-shaped (bathtub curve) for $q < 1$ and $0 < \beta < 1$. The non-monotonic hazard function cited by (Vuorenmaa, 2006) corresponds to the unimodal type and the bathtub shape was not covered by that paper. Figure 2 shows the four possibilities (detailed analysis of the parameters is performed in Section 3), and Figure 3 shows the corresponding four unreliability curves.

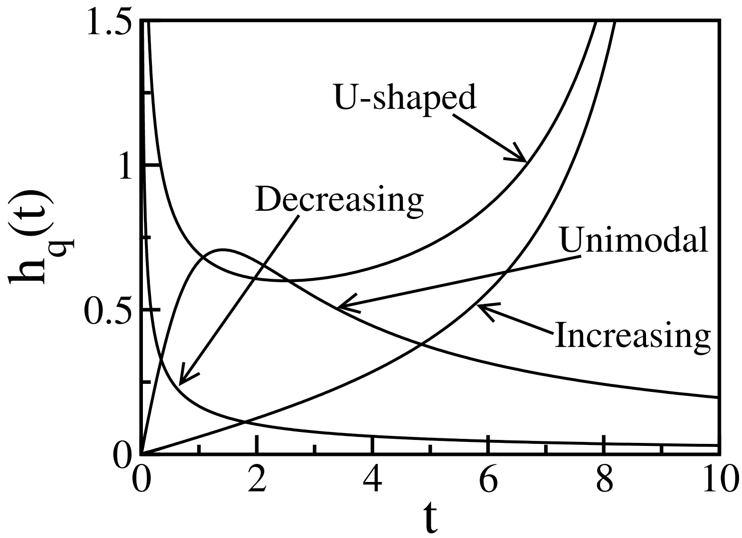

Figure 2 Types of failure rate curves that q-Weibull is able to describe. Values of the parameters were chosen to give a good visualization of the curves in the same figure. The four types are: (i) monotonically decreasing function: $q = 1.5$, $\beta = 0.5$, $\eta = 1$; (ii) monotonically increasing function: $q = 0.5$, $\beta = 2$, $\eta =$

7.071 (evaluated from Eq. (12) with $t_{max} = 10$); (iii) unimodal function: $q = 1.5$, $\beta = 2$, $\eta = 1$; (iv) U-shaped (bathtub curve): $q = 0.5$, $\beta = 0.5$, $\eta = 2.5$ (evaluated from Eq. (12) with $t_{max} = 10$).

For $q < 1$, Eq. (10) presents a divergence that defines the maximum allowed time (lifetime deadline) at

$$t_{max} = t_0 + \left(\eta - t_0\right)(1-q)^{-1/\beta}. \qquad (12)$$

Finite t_{max} corresponds to a relaxation of the constraint usually imposed to a cumulative failure rate function $H_q(t) = \int_0^t h_q(t)dt$ (see (Pham and Lai, 2007)): it is normally expected that $H_1 \to \infty$ at $t \to \infty$. According to q-Weibull model, $H_{q<1} \to \infty$ at $t \to t_{max} < \infty$. That is to say that ordinary Weibull is unlimited, while q-Weibull (with $q < 1$) is limited to t_{max}. Coherently, $\lim_{q \to 1^-} t_{max} \to \infty$.

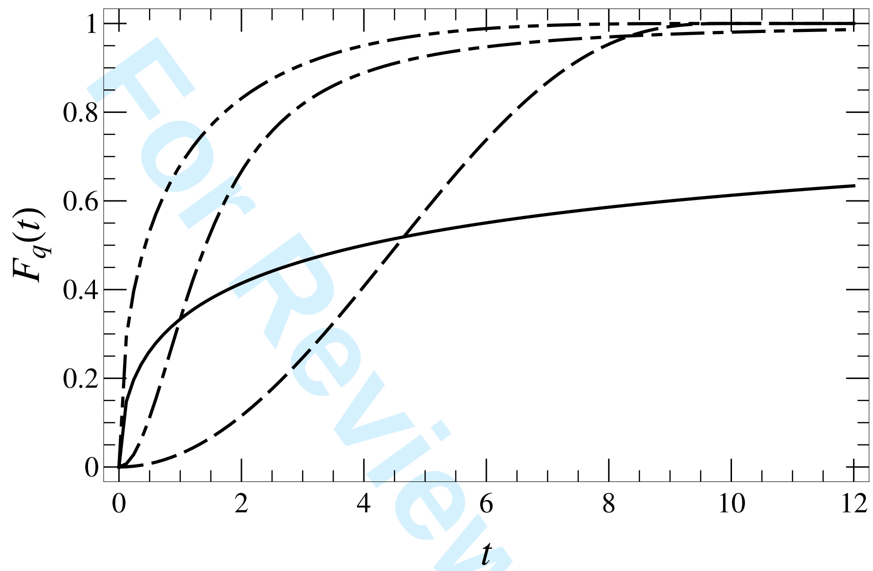

Figure 3 Unreliability curves of q-Weibull distribution. The parameters are the same of the Fig. 2. The four types of failure rate associated are : (i) monotonically decreasing with solid line; (ii) monotonically increasing with dashed line; (iii) unimodal with dot-dashed line; (iv) U-shaped (bathtub curve) with dot-dot-dashed line.

Time derivative of q-failure rate is

$$h'_q(t) = \frac{(2-q)\beta(\beta-1)}{\left(\eta - t_0\right)^2}\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta-2} \times \frac{\left[1-\left(\frac{1-q}{1-\beta}\right)\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]}{\left[1-(1-q)\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]_+^2}. \qquad (13)$$

For the unimodal case ($1 < q < 2$ and $\beta > 1$) and for the U-shaped case ($q < 1$ and $0 < \beta < 1$), the root of Eq. (13) is located at

$$t^* = t_0 + (\eta - t_0)\left(\frac{1-\beta}{1-q}\right)^{1/\beta}, \qquad (14)$$

that corresponds to the extremum value (maximum for unimodal case, minimum for bathtub case)

$$h_q(t^*) = \frac{2-q}{\eta - t_0}\left(\frac{1-\beta}{1-q}\right)^{(\beta-1)/\beta}. \qquad (15)$$

Figure 4 illustrates the change of sign in time derivative of $h_q(t)$.

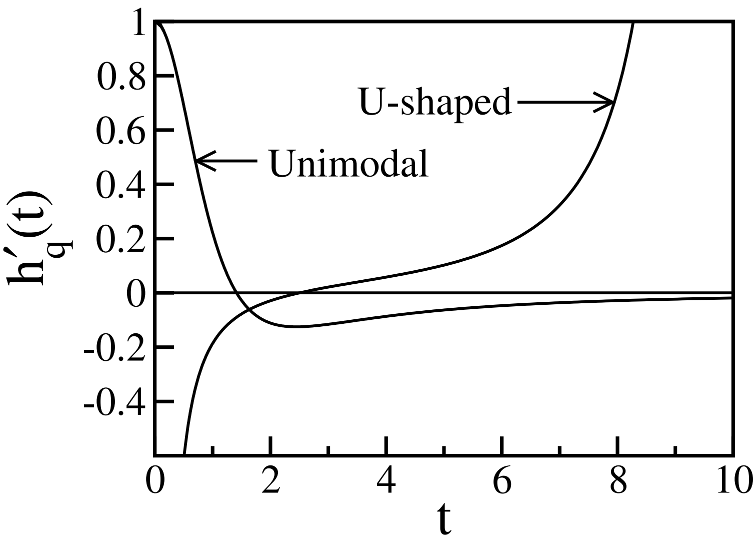

Figure 4 Time derivative $h'_q(t)$, given by Eq. (13), with $q = 0.5$, $\beta = 0.5$ and $\eta = 2.5$ (corresponding to the U-shaped curve) and $q = 1.5$, $\beta = 2$ and $\eta = 1$ (corresponding to the unimodal curve). Parameters are the same of those in Fig. 2 (monotonic cases are not shown). The change of sign in $h'_q(t)$ is responsible for the proper description of the whole bathtub curve.

Time derivative of the usual ($q = 1$) Weibull failure rate is a monotonic power-law,

$$h'_1(t) = \frac{\beta(\beta-1)}{(\eta - t_0)^2}\left(\frac{t-t_0}{\eta - t_0}\right)^{\beta-2}, \qquad (16)$$

hence it is unable to represent the whole bathtub curve. $h'_1(t) < 0$ for $0 < \beta < 1$, and this situation can just describe the *warm in* phase. *Wear out* phase needs $h'_1(t) > 0$, and this happens in usual Weibull for $\beta > 1$. Description of intermediary random failure phase happens by imposing $\beta = 1$. q-Weibull failure rate reproduces the whole curve by a continuous function with the same set of parameters.

3 Behavior of q-Weibull probability distribution function

In this Section we present the moments of q-Weibull pdf and analyze the influence of the parameter q

on the model.

3.1 Moments of *q*-Weibull pdf

In order to evaluate the raw moments (moments about zero) of Eq. (3), $\mu'_n = \int_0^\infty t^n f_q(t)dt$, we shall consider separately the cases $q < 1$ and $q > 1$. It is not necessary to set $\eta = 1$ as shown by (Vuorenmaa, 2006). For the case $q < 1$, it is useful to consider the integral representation of the *q*-exponencial given by (Lenzi et al., 1999). For the case $q > 1$, it is necessary to use the integral representation proposed by (Tsallis, 1994). Straightforward calculations lead to the raw moments. For $q < 1$,

$$\mu'_n = \eta^n \Gamma\left(1+\frac{n}{\beta}\right)\frac{\Gamma\left(\frac{3-2q}{1-q}\right)}{(1-q)^{n/\beta}\Gamma\left(\frac{3-2q}{1-q}+\frac{n}{\beta}\right)}, \quad \text{if } t_0 = 0, \qquad (19)$$

or

$$\mu'_n = \sum_{j=0}^{n}\left\{\begin{array}{c}\binom{n}{j} t_0^{n-j}(\eta - t_0)^j\,\Gamma\left(1+\frac{j}{\beta}\right)\times \\ \frac{\Gamma\left(\frac{3-2q}{1-q}\right)}{(1-q)^{\frac{j}{\beta}}\Gamma\left(\frac{3-2q}{1-q}+\frac{j}{\beta}\right)}\end{array}\right\}, \qquad (20)$$

with $t_0 \neq 0$

and for $q > 1$,

$$\mu'_n = \eta^n \Gamma\left(1+\frac{n}{\beta}\right)\frac{\Gamma\left(\frac{2-q}{q-1}-\frac{n}{\beta}\right)}{(q-1)^{n/\beta}\Gamma\left(\frac{2-q}{q-1}\right)}, \quad \text{if } t_0 = 0, \qquad (21)$$

or

$$\mu'_n = \sum_{j=0}^{n}\left\{\begin{array}{c}\binom{n}{j} t_0^{n-j}(\eta - t_0)^j\,\Gamma\left(1+\frac{j}{\beta}\right)\times \\ \frac{\Gamma\left(\frac{2-q}{q-1}-\frac{j}{\beta}\right)}{(q-1)^{\frac{j}{\beta}}\Gamma\left(\frac{2-q}{q-1}\right)}\end{array}\right\}, \qquad (22)$$

with $t_0 \neq 0$

with $1 < q < q_{upper}$ and $q_{upper} = 1 + \beta/(n+\beta)$. Note that $q \rightarrow 1$ recovers the moments of usual Weibull pdf, $\mu_n' = \eta^n \Gamma\left(1+\frac{n}{\beta}\right)$, for $t_0 = 0$ e $\mu_n' = \sum_{j=0}^{n}\left\{\binom{n}{j} t_0^{n-j}\left(\eta - t_0\right)^j \Gamma\left(1+\frac{j}{\beta}\right)\right\}$, for $t_0 \neq 0$. The upper limit q_{upper} attains the values $\lim_{\beta\to 0} q_{upper} = 1$, $\lim_{\beta\to\infty} q_{upper} = 2$, and $\lim_{n\to\infty} q_{upper} = 1$. The latter limiting behavior means that it is not possible that q-Weibull pdf has all its moments for $q > 1$ (all moments are defined for $q \leq 1$). As q departs from unity from above (for constant β), q-Weibull loses its higher moments (normalizability, that is $\mu'_0 = 1$, is preserved $\forall q < 2$). Note that μ_1' is the mean time between failures (MTBF). There are many distributions that don't have all moments. Cauchy-Lorentz distribution, for instance, has no mean, variance or higher moments. Usual Weibull pdf has all moments, that is typical for distributions with exponential decay.

Central moments (moments about the mean) are found using the binomial transformation of the raw moments, as usual,

$$\mu_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mu_k'\left(\mu_1'\right)^{n-k}, \quad \text{if } t_0 = 0. \qquad (23)$$

or for $t_0 \neq 0$

$$\mu_n = \sum_{j=0}^{n}\left\{\begin{array}{c}\binom{n}{j}\left(t_0 - \mu_1'\right)^{n-j}\left(\eta - t_0\right)^j \times \\ \Gamma\left(1+\frac{j}{\beta}\right)\dfrac{\Gamma\left(\frac{2-q}{q-1} - \frac{j}{\beta}\right)}{(q-1)^{\frac{j}{\beta}}\,\Gamma\left(\frac{2-q}{q-1}\right)}\end{array}\right\}, \qquad (24)$$

$$1 < q < 1 + \frac{\beta}{\beta + n}$$

and

$$\mu_n = \sum_{j=0}^{n}\left\{\begin{array}{c}\binom{n}{j}\left(t_0 - \mu_1'\right)^{n-j}\left(\eta - t_0\right)^j \times \\ \Gamma\left(1+\frac{j}{\beta}\right)\dfrac{\Gamma\left(\frac{3-2q}{1-q}\right)}{(1-q)^{\frac{j}{\beta}}\,\Gamma\left(\frac{3-2q}{1-q} + \frac{j}{\beta}\right)}\end{array}\right\}, \qquad (25)$$

$$q < 1.$$

The median of q-Weibull pdf is $\mathrm{Md} = t_0 + (\eta - t_0)(2^{q'-1} q' \ln_{q'} 2)^{1/\beta}$, with $q' = 1/(2-q)$, and its mode is $\mathrm{Mo} = t_0 + (\eta - t_0)\{(\beta - 1)/[\beta + (1-q)(\beta - 1)]\}^{1/\beta}$, if $\beta > 1$.

3.2 Influence of q

In order to exhibit the effect of the parameter $q < 1$ on the q-Weibull model, let us consider the instance $\beta = 0.5$. Firstly we keep parameter η constant (let us assume $\eta = 1$ for simplicity). The usual

($q = 1$) Weibull does not present a limiting lifetime (i.e., $t_{max} = \infty$). As q departs from unity (from below), lifetime deadline gets smaller values, as Figure 5 depicts. Secondly let us keep t_{max} constant (we choose the instance $\beta = 0.5$ and $t_{max} = 100$), so η is obtained according to Eq. (12). Figure 6 shows curves for different values of q. As q approaches unity (from below), intermediate random failure phase decreases and minimum of failure rate (Eq. (15)) increases. Particularly $\lim_{q\to 1^-} h_q(t^*) \to \infty$. Minimum

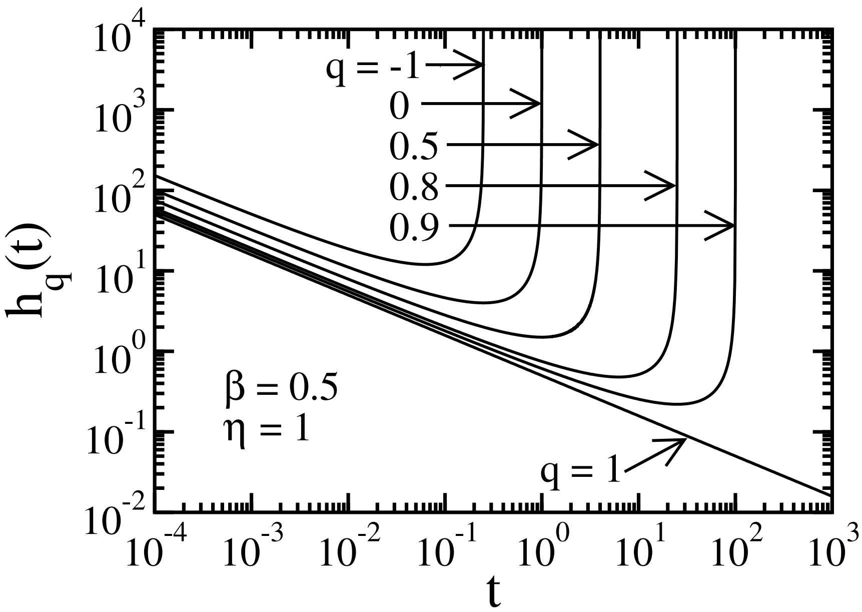

Figure 5 q-Weibull failure rate curve as a function of time for different values of $q < 1$ in log-log scale. All curves are calculated with $\beta = 0.5$ and $\eta = 1$. Limiting lifetime comes from $t = \infty$ for $q = 1$ to closer and finite values as parameter q departs from unity from below.

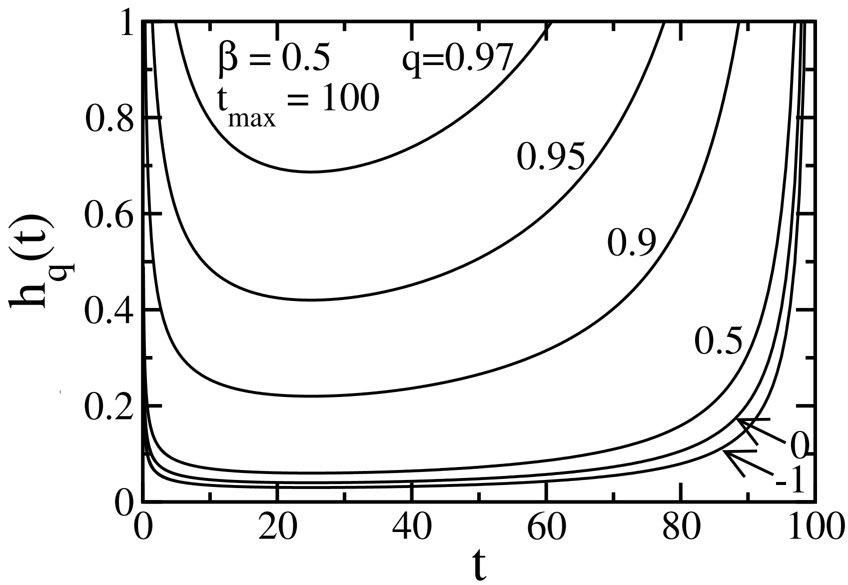

Figure 6 q-Weibull failure rate curve as a function of time for different values of $q < 1$. All curves are calculated with $\beta = 0.5$ and $t_{max} = 100$, so η is taken from Eq. (12): $\eta = 0.09, 0.25, 1, 25, 100, 400$ corresponds to $q = 0.97, 0.95, 0.9, 0.5, 0, -1$ respectively. As q approaches unity, intermediate random failure phase decreases and minimum value of failure rate $h_q(t^*)$ increases ($h_q(t^*) \to \infty$ for

$q \to 1$). As $q \to -\infty$, curves tend to a lower bound (this particular instance, $h_q(t^*) = 0.02$, from Eq. (15) with $\beta = 0.5$ and $t_{max} = 100$).

Influence of q on unimodal case ($1 < q < 2$ and $\beta > 1$) can be viewed in Figure 7. There is a displacement of the maxir

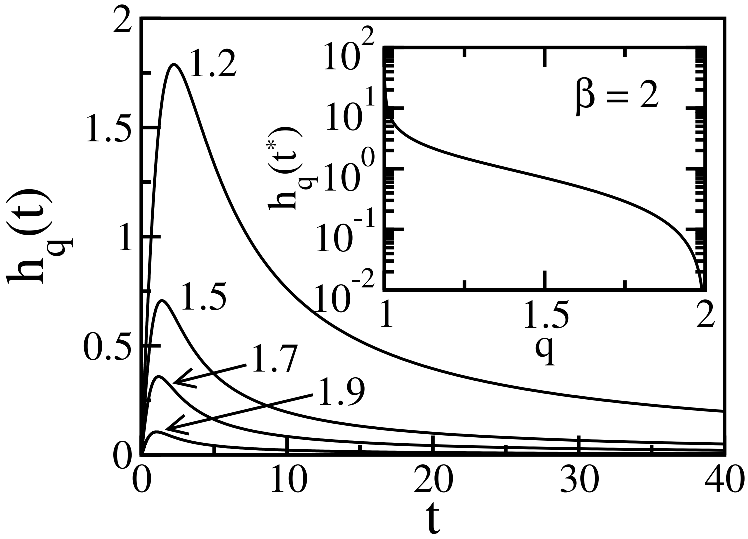

Figure 7 q-Weibull failure rate for unimodal case, with $\beta = 2$ and $\eta = 1$, and different values of $q > 1$ (indicated). Inset shows maximum of failure rate as a function of q (Eq. (15)).

For $1 < q < 2$ and $0 < \beta < 1$, q-Weibull failure rate is a monotonically decreasing function and Figure 8 presents examples.

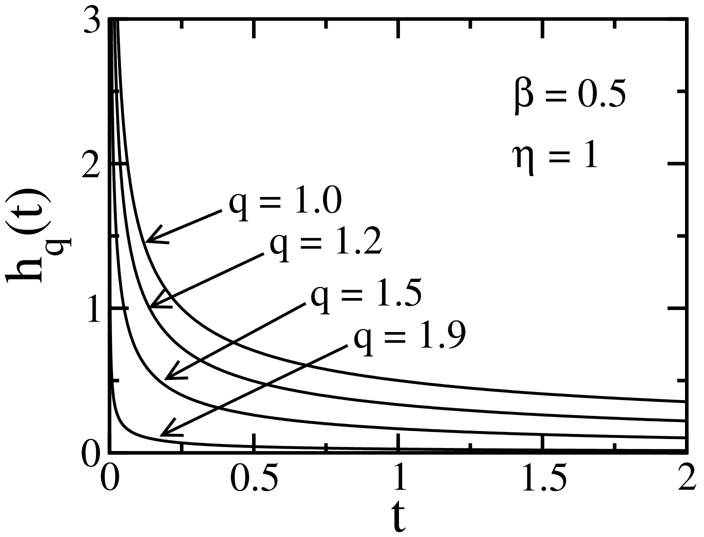

Figure 8 q-Weibull failure rate, given by Eq. (10), with $\beta = 0.5$, $\eta = 1$ and different values of $q > 1$. $h_q(t)$ is a monotonically decreasing function for $\beta < 1$ and $1 \le q < 2$.

4 An example

An example extracted from (Assis et al., 2011) can illustrate the flexibility of q-Weibull distribution in comparison to the usual Weibull model. Time to failure of lower nipple pump data from Brazilian oil wells are collected in days and are shown in Table 1. The maximization of the coefficient of determination R^2 was applied at four different distributions (q-Weibull, Weibull, q-exponential and exponential) in order to calculate the parameters of the distributions in Table 2. Fig. 9 show the results. It is evident that the q-Weibull distribution yields a better fit to the data. Its failure rate shape is a bathtub curve.

Table 1 Nipple pump time to failure, in days in ascending order.

Table 2 Fitting parameters for the Weibull and q-Weibull models, data from Table 1.

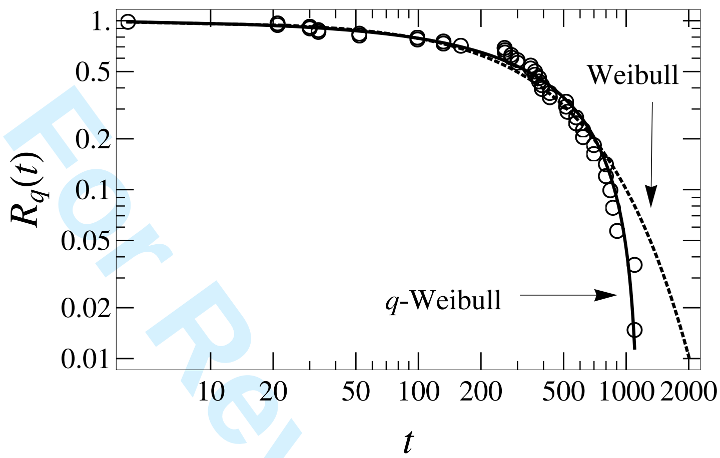

Figure 9 Log-log plot of the data of Table 1 (circles), Weibull (dotted line), q-Weibull (solid line).

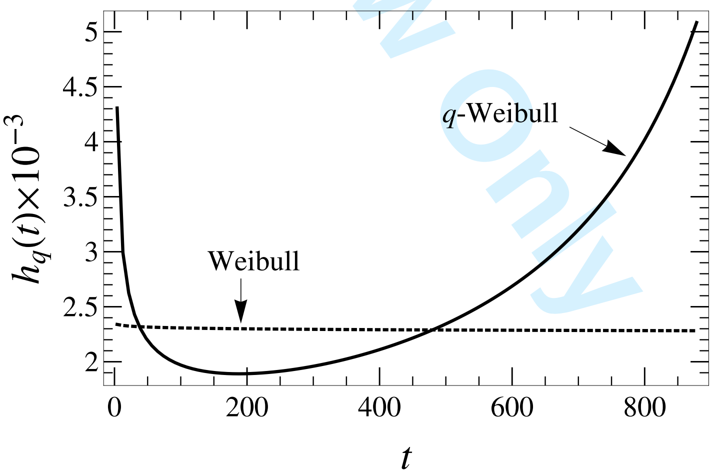

Figure 10 Weibull (dotted line), q-Weibull (solid line) failure rates (data from Table 1).

5 Final remarks

Several models for failure rate function are found in the literature, many of them use Weibull (or Weibull-like) as a basis. These distributions share in common the exponential nature. The q-Weibull generalization uses a function that is exponential only as a limiting case, and may yield asymptotic power-laws. The q-Weibull model is able to describe four types of failure rate function, namely monotonically decreasing, monotonically increasing, unimodal and U-shaped curves, with a single parsimonious set of three parameters, representing a unification of various models, including the versatile Burr XII distribution. Table 3 summarizes the possibilities with the corresponding ranges of parameters.

Table 3 Behavior of q-Weibull failure rate according to the range of parameters q and β.

Usual ($q = 1$) Weibull model is unable to represent the whole bathtub curve, once $h_1(t)$ is monotonically decreasing or monotonically increasing, depending on the value of parameter β. Modeling of U-shaped bathtub curve with Weibull requires a piecewise, discontinuous description with $\beta < 1$ for the *warm in* phase, then $\beta = 1$ for the intermediary random failure phase and finally $\beta > 1$ for the *wear out* phase. q-Weibull continuously reproduces the whole curve with the same set of constant parameters and without need of introducing *ad hoc* hypotheses.

The example given in the text compares the fitting of Weibull and q-Weibull models. In the ordinary Weibull case, the shape parameter is close to one, which represents a constant failure rate (similarly to the exponential distribution). The q-Weibull model provides a better fitting and is able to describe a bathtub shape curve for the failure rate.

q-Weibull is a natural extension of usual Weibull, and it has the advantage of being originated from a theoretical background rooted in nonextensive statistical physics. Of course the introduction of additional (empirically or theoretically based) generalizations, like the use of linear or nonlinear transformation of time, use of multiple distributions, time dependence of parameters, etc., as it was done with Weibull, will further enhance flexibility and accuracy of q-Weibull model.

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Table 1 Nipple pump time to failure, in days in ascending order.

Nipple pump time to failure (days)						
4	21	21	30	30	33	33
52	52	99	99	132	132	160
260	261	261	280	280	300	300
347	347	364	364	381	381	393
393	429	429	514	514	522	574
574	620	620	699	699	799	799
839	863	905	1098	1098		

Table 2 Fitting parameters for the Weibull and q-Weibull models, data from Table 1.

	Weibull	q-Weibull
β	0.99	0.74
η	431	4540
t_0	-3.53	1.48
q	1.00	-1.76
R^2	0.959	0.973

Table 3 Behavior of q-Weibull failure rate according to the range of parameters q and β.

	$0 < \beta < 1$	$\beta = 1$	$\beta > 1$
$q < 1$	bathtub curve	monotonically increasing	monotonically increasing
$q = 1$	monotonically decreasing	constant	monotonically increasing
$1 < q < 2$	monotonically decreasing	monotonically decreasing	unimodal